INDIAN INSTITUTE OF SCIENCE

Water Resources Systems:

Modeling Techniques and Analysis

Lecture - 6

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Summary of the previous lecture

- Constrained optimization

  Maximize or Minimize $f(X)$

  Subject to (s.t.) $g_j(X) \leq 0$ $\quad j = 1, 2, \ldots\ldots m$

- Function with equality constraints

  Maximize or Minimize $f(X)$

  Subject to (s.t.) $g_j(X) = 0$ $\quad j = 1, 2, \ldots\ldots m$

  - Direct substitution

    Reduce the problem to an unconstrained problem by expressing m variables in terms of the remaining $(n - m)$ variables

  - Lagrange multipliers

$$L = f(X) - \sum_{j=1}^{m} \lambda_j g_j(X)$$

Optimization: Methods of Calculus

Necessary condition:

$$L = f(X) - \sum_{j=1}^{m} \lambda_j g_j(X)$$

$$\frac{\partial L}{\partial x_i} = 0 \qquad i = 1, 2, \ldots.. n$$

$$\frac{\partial L}{\partial \lambda_j} = 0 \qquad j = 1, 2, \ldots.. m$$

The $(n + m)$ simultaneous equations are solved to get a solution, (X^*, λ^*) .

Sufficiency condition:

$$L_{ij} = \left.\frac{\partial^2 L}{\partial x_i \partial x_j}\right|_{(X^*, \lambda^*)} \qquad i = 1, 2, \ldots.. n$$

$$g_{ji} = \left.\frac{\partial g_j(X)}{\partial x_i}\right|_{X^*} \qquad j = 1, 2, \ldots.. m$$

Optimization using Calculus

Sufficiency condition:

$$
|D| = \begin{array}{c}
\\
n \text{ terms} \left\{ \begin{array}{c} \\ \\ \\ \\ \\ \end{array} \right. \\
m \text{ terms} \left\{ \begin{array}{c} \\ \\ \\ \\ \\ \end{array} \right.
\end{array}
\overbrace{\hphantom{xx}}^{n \text{ terms}} \overbrace{\hphantom{xx}}^{m \text{ terms}}
\left| \begin{array}{cccc|cccc}
L_{11}-Z & L_{12} & \cdots & L_{1n} & g_{11} & g_{21} & \cdots & g_{m1} \\
L_{21} & L_{22}-Z & \cdots & L_{2n} & g_{12} & g_{22} & \cdots & g_{m2} \\
. & . & & . & . & . & & . \\
. & . & & . & . & . & & . \\
L_{n1} & L_{n2} & \cdots & L_{nn}-Z & g_{1n} & g_{2n} & \cdots & g_{mn} \\
\hline
g_{11} & g_{12} & \cdots & g_{1n} & 0 & 0 & \cdots & 0 \\
g_{21} & g_{22} & \cdots & g_{2n} & 0 & 0 & \cdots & 0 \\
. & . & & . & . & . & . & . \\
. & . & & . & . & . & . & . \\
g_{m1} & g_{m2} & \cdots & g_{mn} & 0 & 0 & \cdots & 0
\end{array} \right|
$$

$$|D| = 0$$

Optimization using Calculus

Leads to a polynomial in Z of the order $(n - m)$

Solve for Z

If all Z values are positive ….. X* corresponds to minimum

If all Z values are negative ….. X* corresponds to maximum

If some values are positive and some are negative … X* is neither a minimum nor a maximum.

Example – 1

Maximize the function,

$$f(X) = -x_1^2 - x_2^2$$

s.t.

$$x_1 + x_2 = 4$$

Solution:

Let

$g(X) = x_1 + x_2 - 4 = 0$

The Lagrange function is

$$L = -x_1^2 - x_2^2 - \lambda(x_1 + x_2 - 4)$$

Example – 1 (Contd.)

Necessary condition: at stationary point,

$$\frac{\partial L}{\partial x_1} = 0; \quad \frac{\partial L}{\partial x_2} = 0; \qquad \frac{\partial L}{\partial \lambda} = 0$$

$$L = -x_1^2 - x_2^2 - \lambda\left(x_1 + x_2 - 4\right)$$

$$\frac{\partial L}{\partial x_1} = 0; \qquad -2x_1 - \lambda = 0$$

$$\frac{\partial L}{\partial x_2} = 0; \qquad -2x_2 - \lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = 0, \qquad -\left(x_1 + x_2 - 4\right) = 0$$

Example – 1 (Contd.)

Solving the equations,

$$-2x_1 - \lambda = 0 \quad \longrightarrow \quad x_1 = -\frac{\lambda}{2}$$

$$-2x_2 - \lambda = 0 \quad \longrightarrow \quad x_2 = -\frac{\lambda}{2}$$

$$-\left(x_1 + x_2 - 4\right) = 0 \qquad -\left(-\frac{\lambda}{4} - \frac{\lambda}{4} - 4\right) = 0$$

$$\lambda = -4$$

Therefore,

$$x_1 = 2, \quad x_2 = 2$$

Example – 1 (Contd.)

Sufficiency condition:

$$\left|D\right| = 0$$

$$\left|D\right| = \begin{vmatrix} L_{11} - Z & L_{12} & g_{11} \\ L_{21} & L_{22} - Z & g_{12} \\ g_{11} & g_{12} & 0 \end{vmatrix} = 0$$

$$L = -x_1^2 - x_2^2 - \lambda\left(x_1 + x_2 - 4\right)$$

$$\frac{\partial L}{\partial x_1} = -2x_1 - \lambda$$

$$L_{11} = \frac{\partial^2 L}{\partial x_1^2} = -2; \qquad L_{12} = \frac{\partial^2 L}{\partial x_1 \partial x_2} = 0$$

Example – 1 (Contd.)

$$\frac{\partial L}{\partial x_2} = -2x_2 - \lambda$$

$$L_{22} = \frac{\partial^2 L}{\partial x_2^2} = -2; \qquad L_{21} = \frac{\partial^2 L}{\partial x_2 \partial x_1} = 0$$

$$g(X) = x_1 + x_2 - 4$$

$$g_{11} = \frac{\partial g}{\partial x_1} = 1; \qquad g_{12} = \frac{\partial g}{\partial x_2} = 1$$

Example – 1 (Contd.)

$$|D| = \begin{vmatrix} -2-Z & 0 & 1 \\ 0 & -2-Z & 1 \\ 1 & 1 & 0 \end{vmatrix} = 0$$

$$2Z + 4 = 0$$

$$Z = -2$$ only one root

As the root is negative, the stationary point X=(2,2) is a local maximum of $f(X)$ and $f_{max}(X) = 8$

Example – 2

Minimize the function

$$f(X) = \frac{1}{2}\left[x_1^2 + x_2^2 + x_3^2\right]$$

s.t.

$$x_1 - x_2 = 0$$

$$x_1 + x_2 + x_3 = 1$$

Solution:

Let $g_1(X) = x_1 - x_2 = 0$

$$g_2(X) = x_1 + x_2 + x_3 - 1 = 0$$

Exercise problem no.2.49 from "Engineering and optimization-theory and practice" by Singiresu S. Rao, 1996, John Wiley & Sons

Example – 2 (Contd.)

Lagrange function is

$$L = \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2\right) + \lambda_1\left(x_1 - x_2\right) + \lambda_2\left(x_1 + x_2 + x_3 - 1\right)$$

Necessary condition:

$$\frac{\partial L}{\partial x_i} = 0 \qquad \forall \quad i$$

$$\frac{\partial L}{\partial \lambda_j} = 0 \qquad \forall \quad j$$

Example – 2 (Contd.)

$$L = \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2\right) + \lambda_1\left(x_1 - x_2\right) + \lambda_2\left(x_1 + x_2 + x_3 - 1\right)$$

$$\frac{\partial L}{\partial x_1} = 0; \qquad x_1 + \lambda_1 + \lambda_2 = 0$$

$$\frac{\partial L}{\partial x_3} = 0; \qquad x_3 + \lambda_2 = 0$$

$$\frac{\partial L}{\partial x_2} = 0; \qquad x_2 - \lambda_1 + \lambda_2 = 0$$

$$\left.\begin{aligned} \frac{\partial L}{\partial \lambda_1} = 0; &\qquad x_1 - x_2 = 0 \\ \frac{\partial L}{\partial \lambda_2} = 0; &\qquad x_1 + x_2 + x_3 - 1 = 0 \end{aligned}\right\}$$

Original constraint equations

Example – 2 (Contd.)

$$\left.\begin{aligned} x_1 + \lambda_1 + \lambda_2 &= 0 \\ x_2 - \lambda_1 + \lambda_2 &= 0 \\ x_1 + x_2 + x_3 - 1 &= 0 \\ x_3 + \lambda_2 &= 0 \\ x_1 - x_2 &= 0 \end{aligned}\right\}$$

5 equations,
5 unknowns

Solving the equations,

$$x_1 = \frac{1}{3}; \quad x_2 = \frac{1}{3}; \quad x_3 = \frac{1}{3}; \quad \lambda_2 = -\frac{1}{3}; \qquad \lambda_1 = 0$$

$$X^* = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} \qquad \lambda^* = \begin{bmatrix} 0 \\ -1/3 \end{bmatrix}$$

Example – 2 (Contd.)

Sufficiency condition:

$$|D| = \begin{vmatrix} L_{11} - Z & L_{12} & L_{13} & g_{11} & g_{21} \\ L_{21} & L_{22} - Z & L_{23} & g_{12} & g_{22} \\ L_{31} & L_{32} & L_{33} - Z & g_{13} & g_{23} \\ g_{11} & g_{12} & g_{13} & 0 & 0 \\ g_{21} & g_{21} & g_{23} & 0 & 0 \end{vmatrix} = 0$$

$$\frac{\partial L}{\partial x_1} = x_1 + \lambda_1 + \lambda_2$$

$$L_{11} = \frac{\partial^2 L}{\partial x_1^2} = 1 \quad ; \quad L_{12} = \frac{\partial^2 L}{\partial x_1 \partial x_2} = 0 \quad ; \quad L_{13} = \frac{\partial^2 L}{\partial x_1 \partial x_3} = 0$$

Example – 2 (Contd.)

$$\frac{\partial L}{\partial x_2} = x_2 - \lambda_1 + \lambda_2$$

$$L_{21} = \frac{\partial^2 L}{\partial x_2 \partial x_1} = 0 \quad ; \qquad L_{22} = \frac{\partial^2 L}{\partial x_2^2} = 1 \quad ; \qquad L_{23} = \frac{\partial^2 L}{\partial x_2 \partial x_3} = 0$$

$$\frac{\partial L}{\partial x_3} = x_3 + \lambda_2$$

$$L_{31} = \frac{\partial^2 L}{\partial x_3 \partial x_1} = 0 \quad ; \qquad L_{32} = \frac{\partial^2 L}{\partial x_3 \partial x_2} = 0 \quad ; \qquad L_{33} = \frac{\partial^2 L}{\partial x_3^2} = 1$$

Example – 2 (Contd.)

$$g_1(X) = x_1 - x_2 = 0$$

$$g_{11} = \frac{\partial g_1}{\partial x_1} = 1 \quad ; \qquad g_{12} = \frac{\partial g_1}{\partial x_2} = -1 \quad ; \qquad g_{13} = \frac{\partial g_1}{\partial x_3} = 0$$

$$g_2(X) = x_1 + x_2 + x_3 - 1 = 0$$

$$g_{21} = \frac{\partial g_2}{\partial x_1} = 1 \quad ; \qquad g_{22} = \frac{\partial g_2}{\partial x_2} = 1 \quad ; \qquad g_{23} = \frac{\partial g_2}{\partial x_3} = 1$$

Example – 2 (Contd.)

$$|D| = \begin{vmatrix} 1-Z & 0 & 0 & 1 & 1 \\ 0 & 1-Z & 0 & -1 & 1 \\ 0 & 0 & 1-Z & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{vmatrix} = 0$$

This is a polynomial in Z of the order $(n - m)$

$= (3 - 2)$

$= 1$

$$6(1-Z) = 0$$

$$Z = 1 \qquad > 0$$

Example – 2 (Contd.)

As the root is positive, the stationary point

$$X^* = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$ is a local minimum of *f(X)*

and $$f_{\min}(X) = \frac{1}{2}\left[x_1^2 + x_2^2 + x_3^2\right]$$

$$= \frac{1}{2}\left[\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2\right]$$

$$= \frac{1}{6}$$

Kuhn – Tucker Conditions

Optimization: Methods of Calculus

Minimize $f(X)$

s.t.

$g_j(X) \leq 0 \qquad j = 1, 2, \ldots.. m$

Kuhn – Tucker conditions:

Get (n+m) equations and solve

$$\frac{\partial f}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_i} = 0 \qquad i = 1, 2, \ldots.. n$$

$$\lambda_j g_j = 0 \qquad j = 1, 2, \ldots.. m$$

Check for validity

$$g_j \leq 0 \qquad j = 1, 2, \ldots.. m$$

and $\lambda_j \geq 0 \qquad j = 1, 2, \ldots.. m$