INDIAN INSTITUTE OF SCIENCE

Water Resources Systems:

Modeling Techniques and Analysis

Lecture - 11

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Summary of the previous lecture

- Algebraic approach for LP solution – Simplex algorithm
  - Standard form of LP
  - Initial basic feasible solution
  - Entering variable; departing variable
  - Optimality condition
- Multiple solutions
  - Non-basic variable in the final tableau, with a coefficient of zero in the Z-row.

Max $Z = 3x_1 + 5x_2$

$x_1 \leq 4$

$2x_2 \leq 12$

$3x_1 + 2x_2 \leq 18$

$x_1 \geq 0$

$x_2 \geq 0$

Final Tableau (Optimal solution)

Basis	Row	Z	x_1	x_2	x_3	x_4	x_5	b_i
Z	0	1	0	0	0	3/2	1	36
x_3	1	0	0	0	1	1/3	-1/3	2
x_2	2	0	0	1	0	1/2	0	6
x_1	3	0	1	0	0	-1/3	1/3	2

Example – 1

Maximize

$$Z = 40x_1 + 100x_2$$ Objective function

s.t.

$$10x_1 + 5x_2 \leq 2500$$
$$4x_1 + 10x_2 \leq 2000$$
$$2x_1 + 3x_2 \leq 900$$

} Constraints

$$x_1 \geq 0$$
$$x_2 \geq 0$$

} Non-negativity of decision variables

Example – 1 (Contd.)

The problem is converted to standard LP form

Maximize $Z = 40x_1 + 100x_2 + 0x_3 + 0x_4 + 0x_5$

s.t.

$$10x_1 + 5x_2 \le 2500 \longrightarrow 10x_1 + 5x_2 + x_3 = 2500$$

$$4x_1 + 10x_2 \le 2000 \longrightarrow 4x_1 + 10x_2 + x_4 = 2000$$

$$2x_1 + 3x_2 \le 900 \longrightarrow 2x_1 + 3x_2 + x_5 = 900$$

$$x_1 \ge 0 \qquad\qquad x_1 \ge 0; \quad x_2 \ge 0$$

$$x_2 \ge 0 \qquad\qquad x_3 \ge 0; \quad x_4 \ge 0; \quad x_5 \ge 0$$

n= no. of variables = 5;   m = no. of constraints = 3

Example – 1 (Contd.)

Iteration-1

Entering variable: x_2

Departing variable: x_4

Basis	Row	Z	x_1	x_2	x_3	x_4	x_5	b_i	b_i/a_{ij}
Z	0	1	-40	-100	0	0	0	0	–
x_3	1	0	10	5	1	0	0	2500	500
x_4	2	0	4	10	0	1	0	2000	200
x_5	3	0	2	3	0	0	1	900	300

Pivot point: 10 (row x_4, column x_2)

$0 - \frac{(-100 \times 2000)}{10}$

$2500 - \frac{(5 \times 2000)}{10}$

Example – 1 (Contd.)

Iteration-2

Basis	Row	Z	x_1	x_2	x_3	x_4	x_5	b_i
Z	0	1	0	0	0	10	0	20,000
x_3	1	0	8	0	1	-1/2	0	1500
x_2	2	0	4/10	1	0	1/10	0	200
x_5	3	0	8/10	0	0	-3/10	1	300

Example – 1 (Contd.)

Optimal solution; since all coefficients in Z-row are non-negative.

$$Z = 20,000$$

$$x_1 = 0 \longrightarrow$$ Coefficient of x_1 (a non basic variable) is zero

$$x_2 = 200$$

$$x_3 = 1500$$

$$x_4 = 0; \quad x_5 = 300$$

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Multiple solutions exist

Make x_1 the entering variable

Example – 1 (Contd.)

Iteration-2

Entering variable: x_1

Departing variable: x_3

Basis	Row	Z	x_1	x_2	x_3	x_4	x_5	b_i	b_i/a_{ij}
Z	0	1	0	0	0	10	0	20,000	–
x_3	1	0	8	0	1	-1/2	0	1500	1500/8
x_2	2	0	4/10	1	0	1/10	0	200	500
x_5	3	0	8/10	0	0	-3/10	1	300	375

Pivot point: 8

Example – 1 (Contd.)

Iteration-3

Basis	Row	Z	x_1	x_2	x_3	x_4	x_5	b_i
Z	0	1	0	0	0	10	0	20,000
x_1	1	0	1	0	1/8	-1/16	0	1500/8
x_2	2	0	0	1	-1/20	1/8	0	125
x_5	3	0	0	0	-1/10	-1/4	1	150

Example – 1 (Contd.)

Another solution to the problem is

$$Z = 20{,}000$$

$$x_1 = 1500/8$$

$$x_2 = 125$$

$$x_3 = 0; \quad x_4 = 0;$$

$$x_5 = 150$$

Two sets of optimal solution X_1 and X_2

$$X_1 = \begin{bmatrix} 0 \\ 200 \\ 1500 \\ 0 \\ 300 \end{bmatrix} \qquad X_2 = \begin{bmatrix} 1500/8 \\ 125 \\ 0 \\ 0 \\ 150 \end{bmatrix}$$

Example – 1 (Contd.)

$$X^* = \alpha X_1 + (1-\alpha) X_2$$

$0 \leq \alpha \leq 1$ is also an optimal solution

$$X^* = \begin{bmatrix} 0 + (1-\alpha)\times 1500/8 \\ 200\alpha + (1-\alpha)\times 125 \\ 1500\alpha + 0 \\ 0+0 \\ 300\alpha + (1-\alpha)\times 150 \end{bmatrix} = \begin{bmatrix} (1-\alpha)\times 1500/8 \\ 125 + 75\alpha \\ 1500\alpha \\ 0 \\ 150 + 150\alpha \end{bmatrix}$$

Example – 1 (Contd.)

For example, α = 0.5 will give

$$x_1 = 93.75$$

$$x_2 = 162.5$$

$$x_3 = 750$$

$$x_4 = 0$$

$$x_5 = 225$$

$$Z = 40x_1 + 100x_2$$

$$= 20,000$$

Z remains same for all solutions

LP – Artificial Variables

- Artificial variables (Big M method):
  - In case of = and $\geq$ constraints, artificial variables are added.
  - Add artificial variable to constraint.
  - Penalize artificial variable in the objective function.
  - Modify row-0 with

$$\widehat{E} = E - \frac{R \times C}{P}$$

  assuming the column of artificial variable as pivotal column and the constraint containing the artificial variable as pivotal row.

Example – 2

Maximize

$$Z = 3x_1 + 5x_2$$

s.t.

$$x_1 \le 4$$
$$2x_2 \le 12$$
$$3x_1 + 2x_2 = 18$$

Constraints

$$x_1 \ge 0$$
$$x_2 \ge 0$$

Non-negativity of decision variables

Big M A_1

$x_1 + x_3 = 4$

$2x_2 + x_4 = 12$

$3x_1 + 2x_2 = 18$

$3x_1 + 2x_2 + A_1 = 18$

Artificial Variable

Example – 2 (Contd.)

The problem is converted to standard LP form

Maximize $Z = 3x_1 + 5x_2$

s.t.

$$x_1 \le 4 \longrightarrow x_1 + x_3 = 4$$

$$2x_2 \le 12 \longrightarrow 2x_2 + x_4 = 12$$

$$3x_1 + 2x_2 = 18 \longrightarrow 3x_1 + 2x_2 = 18$$

$$x_1 \ge 0 \qquad x_1 \ge 0; \; x_2 \ge 0$$

$$x_2 \ge 0 \qquad x_3 \ge 0; \; x_4 \ge 0$$

n= no. of variables = 4; m = no. of constraints =2

Example – 2 (Contd.)

No initial basic feasible solution is available for this problem.

Add artificial variable to constraint 3

$$Z - 3x_1 - 5x_2 + M \times A_1 = 0$$

$$3x_1 + 2x_2 + A_1 = 18$$

Transformation of coefficients in row-0

Example – 2 (Contd.)

$$Z - 3x_1 - 5x_2 + M \times A_1 = 0$$

$$3x_1 + 2x_2 + A_1 = 18$$

$$0 - \frac{M \times 18}{1}$$

x_1	x_2	x_3	x_4	A	b_i
-3	-5	0	0	M	0
3	2	0	0	1	18

$$-3 - \frac{M \times 3}{1}$$

$$\hat{E} = E - \frac{R \times C}{P}$$

R = M; P = 1

-3M-3	-2M-5	0	0	0	-18M