INDIAN INSTITUTE OF SCIENCE

Water Resources Systems:

Modeling Techniques and Analysis

Lecture - 12

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Summary of the previous lecture

- Multiple solutions
    - Non-basic variable in the final tableau, with a coefficient of zero in the Z-row.

- Artificial variables
    - To ensure an initial basic feasible solution.
    - Add artificial variable to constraints of the type $\geq$ and =.
    - Penalize artificial variable in the OF.
    - Modify Row-0 with

$$\widehat{E} = E - \frac{R \times C}{P}$$

Example – 1

Maximize

$$Z = 3x_1 + 5x_2$$

s.t.

$$x_1 \le 4$$

$$2x_2 \le 12$$

$$3x_1 + 2x_2 \overset{\le}{=} 18$$

$$x_1 \ge 0$$

$$x_2 \ge 0$$

$Z = 3x_1 + 5x_2 + 0x_3 + 0x_4 - MA_1$

$3x_1 + 2x_2 + A_1 = 18$

Opt $Z = 36$

Example – 1 (Contd.)

$$Z - 3x_1 - 5x_2 + M \times A_1 = 0$$

$$3x_1 + 2x_2 + A_1 = 18$$

	x_1	x_2	x_3	x_4	A_1	b_i
Row-0	-3	-5	0	0	M	0
Row containing artificial variables	3	2	0	0	1	18

$$\widehat{E} = E - \frac{R \times C}{P}$$

R = M; P = 1

Transformed Row-0	-3M-3	-2M-5	0	0	0	-18M

Example – 1 (Contd.)

- In case of multiple artificial variables, carryout the transformation one by one.
- Use the transformed Z-row in the initial simplex table.

Example – 1 (Contd.)

$= -18M - \frac{(-3M-3)\times 4}{1}$

Iteration-1

Entering variable

Departing variable

Basis	Row	x_1	x_2	x_3	x_4	A_1	b_i	b_i/a_{ij}
Z	0	-3M-3	-2M-5	0	0	0	-18M	–
x_3	1	1	0	1	0	0	4	4
x_4	2	0	2	0	1	0	12	–
A_1	3	3	2	0	0	1	18	6

Pivot point

~~$-18M - (-12M - 3)$~~

Example – 1 (Contd.)

Iteration-2

Entering variable

Departing variable

Basis	Row	x_1	x_2	x_3	x_4	A_1	b_i	b_i/a_{ij}
Z	0	0	-2M-5	3M+3	0	0	-6M+12	–
x_1	1	1	0	1	0	0	4	–
x_4	2	0	2	0	1	0	12	6
A_1	3	0	2	-3	0	1	6	3

Example – 1 (Contd.)

Iteration-3

Entering variable

Departing variable

Basis	Row	x_1	x_2	x_3	x_4	A_1	b_i	b_i/a_{ij}
Z	0	0	0	-9/2	0	M+5/2	27	–
x_1	1	1	0	1	0	0	4	4
x_4	2	0	0	3	1	-1	6	2
x_2	3	0	1	-3/2	0	1/2	3	–

Example – 1 (Contd.)

Iteration-4

Basis	Row	Z	x_1	x_2	x_3	x_4	A_1	b_i
Z	0	1	0	0	0	3/2	M+1	36
x_1	1	0	1	0	0	-1/3	1/3	2
x_3	2	0	0	0	1	1/3	-1/3	2
x_2	3	0	0	1	0	1/2	0	6

Example – 1 (Contd.)

Since all coefficients in the Z-row are non-negative this is the optimal solution.

$$Z = 36$$

$$x_1 = 2$$

$$x_2 = 6$$

$$x_3 = 2$$

$$x_4 = 0$$

$$A_1 = 0$$

Note that this is the same solution as obtained earlier, with the constraint

$$3x_1 + 2x_2 \leq 18$$

↓

Binding (tight) constraint

LP – Unbounded Solution

Unbounded solution:

- No departing variable can be found at some iteration
    - Implies that Z can be increased indefinitely without violating any constraint

Example – 2

Maximize

$$Z = x_1 + x_2$$

s.t.

$$x_1 \geq 5$$

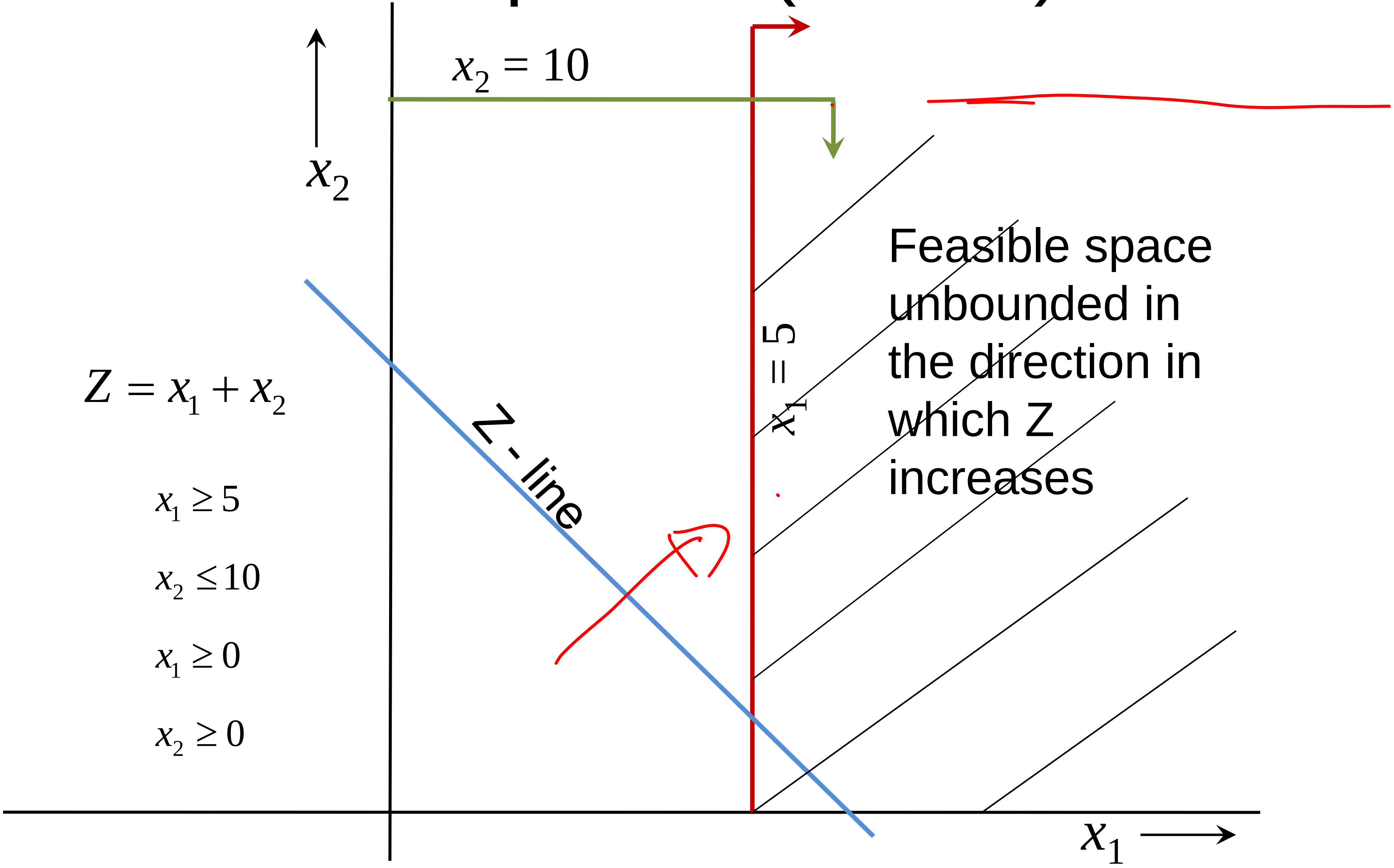
$$x_2 \leq 10$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

x_1 tends to infinity, therefore Z tends to infinity;
hence the problem is unbounded.

Example – 2 (Contd.)



$Z = x_1 + x_2$

$x_1 \geq 5$

$x_2 \leq 10$

$x_1 \geq 0$

$x_2 \geq 0$

Feasible space unbounded in the direction in which Z increases

Example – 2 (Contd.)

The problem is converted to standard LP form

Maximize $Z = x_1 + x_2$

s.t.

$$x_1 \geq 5 \longrightarrow x_1 - x_3 = 5$$

$$x_2 \leq 10 \longrightarrow x_2 + x_4 = 10$$

$$x_1 \geq 0 \qquad x_1 \geq 0;\ \ x_2 \geq 0$$

$$x_2 \geq 0 \qquad x_3 \geq 0;\ \ x_4 \geq 0$$

Example – 2 (Contd.)

Add artificial variable to constraint 1

$$Z - x_1 - x_2 + M \times A_1 = 0$$

$$x_1 - x_3 + A_1 = 5$$

Transformation of coefficients in Row-0

x_1	x_2	x_3	x_4	A	b_i
-1	-1	0	0	M	0
1	0	-1	0	1	5
-1-M	-1	M	0	0	-5M

Example – 2 (Contd.)

Iteration-1

Entering variable

Departing variable

Basis	Row	x_1	x_2	x_3	x_4	A_1	b_i	b_i/a_{ij}
Z	0	-1-M	-1	M	0	0	-5M	–
A_1	1	1	0	-1	0	1	5	5
x_4	2	0	1	0	1	0	10	–

Pivot point

Example – 2 (Contd.)

Iteration-2

Entering variable

Departing variable

Basis	Row	x_1	x_2	x_3	x_4	A_1	b_i	b_i/a_{ij}
Z	0	0	-1	-1	0	1+M	5	–
x_1	1	1	0	-1	0	1	5	–
x_4	2	0	1	0	1	0	10	10

Pivot point

Example – 2 (Contd.)

Iteration-3

Entering variable ↓ x_3

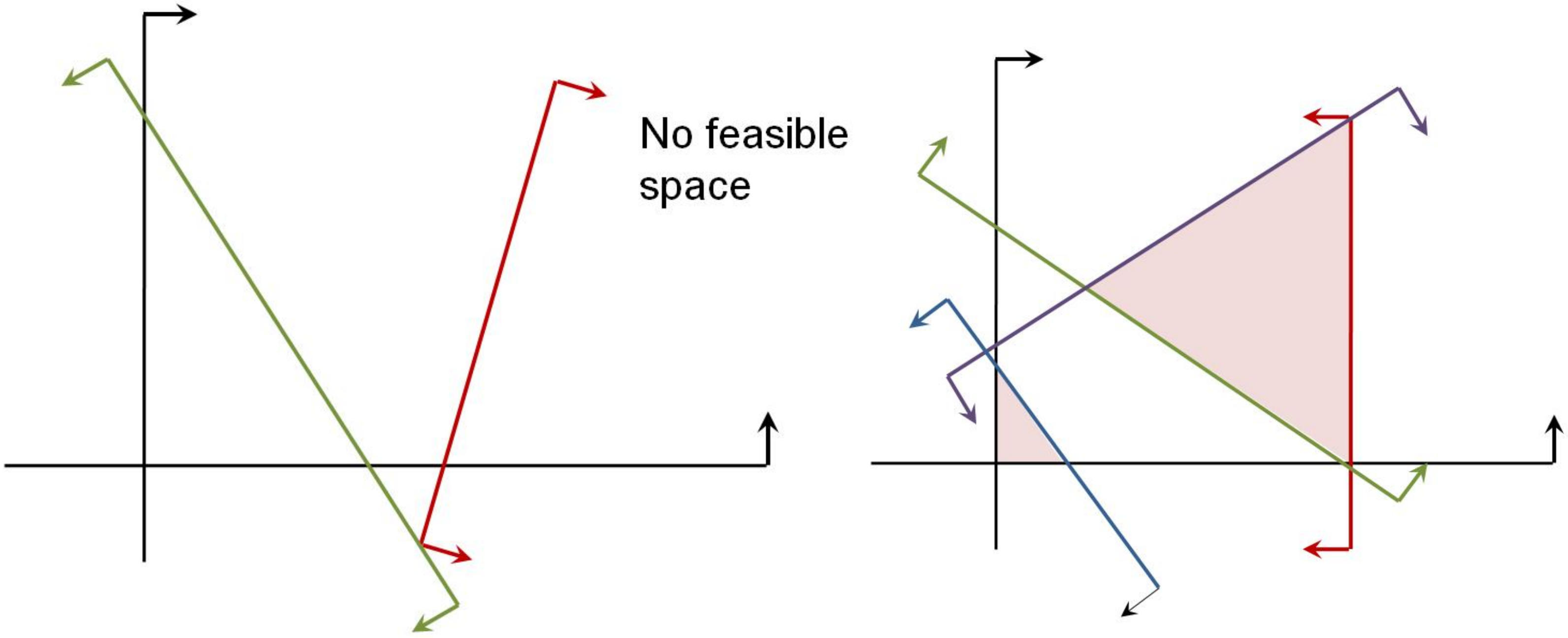
Basis	Row	x_1	x_2	x_3	x_4	A_1	b_i	b_i/a_{ij}
Z	0	0	0	-1	1	1+M	-5	–
x_1	1	1	0	-1	0	1	5	–
x_2	2	0	1	0	1	0	10	–

No departing variable
Hence the problem is unbounded

LP – Infeasible Solution

Infeasible solution:

- One or more artificial variables remain in the basis even when optimality criterion is satisfied.



Example – 3

Minimize

$$Z = 3x_1 + 5x_2$$

s.t.

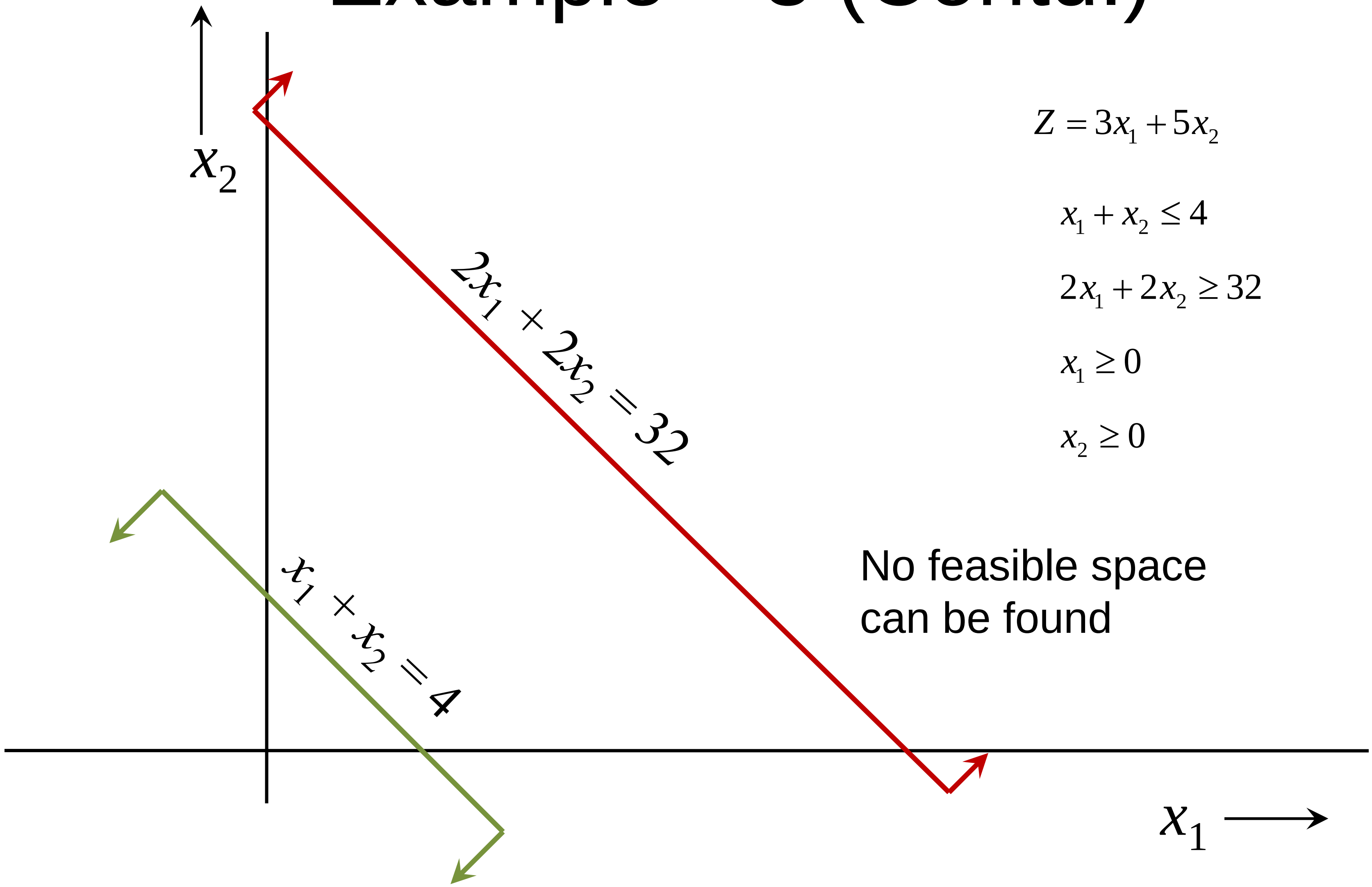
$$x_1 + x_2 \leq 4$$

$$2x_1 + 2x_2 \geq 32$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

Example – 3 (Contd.)



$$Z = 3x_1 + 5x_2$$

$$x_1 + x_2 \le 4$$

$$2x_1 + 2x_2 \ge 32$$

$$x_1 \ge 0$$

$$x_2 \ge 0$$

No feasible space can be found

Example – 3 (Contd.)

The problem is converted to standard LP form

Maximize $Z = 3x_1 + 5x_2$

s.t.

$$x_1 + x_2 \le 4 \longrightarrow x_1 + x_2 + x_3 = 4$$

$$2x_1 + 2x_2 \ge 32 \longrightarrow 2x_1 + 2x_2 - x_4 = 32$$

$$x_1 \ge 0 \qquad x_1 \ge 0; \quad x_2 \ge 0$$

$$x_2 \ge 0 \qquad x_3 \ge 0; \quad x_4 \ge 0$$

Example – 3 (Contd.)

Add artificial variable to constraint 2

$$Z - 3x_1 - 5x_2 + M \times A_1 = 0$$

$$2x_1 + 2x_2 - x_4 + A_1 = 32$$

Transformation of coefficients in Row-0

x_1	x_2	x_3	x_4	A	b_i
-3	-5	0	0	M	0
2	2	0	-1	1	32
-3-2M	-5-2M	0	M	0	-32M

Example – 3 (Contd.)

Iteration-1

Entering variable

Departing variable

Basis	Row	x_1	x_2	x_3	x_4	A_1	b_i	b_i/a_{ij}
Z	0	-3-2M	-5-2M	0	M	0	-32M	–
x_3	1	1	1	1	0	0	4	4
A_1	2	2	2	0	-1	1	32	16

Pivot point

Example – 3 (Contd.)

Iteration-2

Basis	Row	x_1	x_2	x_3	x_4	A_1	b_i
Z	0	2	0	5+2M	M	0	20-24M
x_2	1	1	1	1	0	0	4
A_1	2	0	0	-2	-1	1	24

All coefficients in the Z-row are non-negative

Artificial variable still remains in the basis; Therefore the problem is infeasible.