INDIAN INSTITUTE OF SCIENCE

Water Resources Systems:
Modeling Techniques and Analysis

Lecture - 13

Course Instructor : Prof. P. P. MUJUMDAR
Department of Civil Engg., IISc.

Summary of the previous lecture

Simplex Algorithm: :

- Example on use of artificial variables
- Unbounded solution
  - No departing variable can be found at some iteration
- Infeasible solution
  - One or more artificial variables remain in the basis even when optimality criterion is satisfied

LP – DUAL PROBLEM

LP – Dual Problem

PRIMAL

Maximize $Z_x = 6x_1 + 8x_2$

s.t.

$5x_1 + 10x_2 \le 60$ ….. y_1

$4x_1 + 4x_2 \le 40$ ….. y_2

$x_1 \ge 0$

$x_2 \ge 0$

DUAL

Min $Z_y = 60y_1 + 40y_2$

s.t.

$5y_1 + 4y_2 \geq 6$

$10y_1 + 4y_2 \geq 8$

$y_1 \geq 0; \; y_2 \geq 0$

LP – Dual Problem

Primal problem

Max $Z_x = 6x_1 + 8x_2$

s.t.

$$5x_1 + 10x_2 \leq 60$$
$$4x_1 + 4x_2 \leq 40$$

$$x_1 \geq 0; \quad x_2 \geq 0$$

Dual problem

Min $Z_y = 60y_1 + 40y_2$

s.t.

$$5y_1 + 4y_2 \geq 6$$
$$10y_1 + 4y_2 \geq 8$$

$$y_1 \geq 0; \quad y_2 \geq 0$$

LP – Dual Problem

- Formulation of dual from primal:
  - Express the primal as

    Maximization problem

    with $\leq$ or $=$ constraints

    or

    Minimization problem

    with $\geq$ or $=$ constraints

LP – Dual Problem

Primal		Dual	
OF: Maximize		Minimize	
Variable i		Constraint i	
j^{th} constraint		j^{th} variable	
$x_i \geq 0$		i^{th} constraint	'$\geq$'
x_i unrestricted		i^{th} constraint	'='
j^{th} constraint	'$\leq$'	j^{th} variable	'≥ 0'
j^{th} constraint	'='	j^{th} variable unrestricted	

LP – Dual Problem

Primal	Dual
RHS of constraints (b_i)	Coefficients in objective function
Coefficients in objective function	RHS of constraints

LP – Dual Problem

e.g.,

Maximize $Z = 2x_1 + x_2$

s.t.

$$x_1 - 2x_2 \geq 2$$

$$x_1 + 2x_2 = 8$$

$$x_1 - x_2 \leq 11$$

$$x_1 \geq 0$$

x_2 unrestricted

LP – Dual Problem

The problem is rewritten as

Minimize $Z^{'} = -2x_1 - x_2$

s.t.

$$x_1 - 2x_2 \geq 2 \qquad \ldots\ldots \quad y_1$$

$$x_1 + 2x_2 = 8 \qquad \ldots\ldots \quad y_2$$

$$-x_1 + x_2 \geq -11 \qquad \ldots\ldots \quad y_3$$

$x_1 \geq 0;\ x_2$ unrestricted

LP – Dual Problem

Dual of the problem

Maximize $2y_1 + 8y_2 - 11y_3$

s.t.

$$1 \times y_1 + 1 \times y_2 - 1 \times y_3 \leq -2$$

$$-2 \times y_1 + 2 \times y_2 + 2 \times y_3 = -1$$

$y_1 \geq 0;\ y_2$ unrestricted

$y_3 \geq 0$

Because x_2 is unrestricted

Because second constraint of primal is '='

$$Z' = -2x_1 - x_2$$

$$x_1 - 2x_2 \geq 2$$

$$x_1 + 2x_2 = 8$$

$$-x_1 + x_2 \geq -11$$

$x_1 \geq 0;\ x_2$ unrestricted

Example – 4

Solve the dual of the problem

Maximize

$$Z = 3x_1 + 5x_2$$

s.t.

$$x_1 \le 4$$
$$2x_2 \le 12$$
$$3x_1 + 2x_2 \le 18$$
$$x_1 \ge 0$$
$$x_2 \ge 0$$

Example – 4 (Contd.)

Primal problem

Maximize $Z = 3x_1 + 5x_2$

s.t.

$x_1 \leq 4$ ….. y_1

$2x_2 \leq 12$ ….. y_2

$3x_1 + 2x_2 \leq 18$ ….. y_3

$x_1 \geq 0; \quad x_2 \geq 0$

Example – 4 (Contd.)

Primal problem

Max $Z = 3x_1 + 5x_2$

s.t.

$$x_1 \le 4$$
$$2x_2 \le 12$$
$$3x_1 + 2x_2 \le 18$$

$$x_1 \ge 0; \quad x_2 \ge 0$$

Dual problem

Min $Z' = 4y_1 + 12y_2 + 18y_3$

s.t.

$$y_1 + 0y_2 + 3y_3 \ge 3$$
$$0y_1 + 2y_2 + 2y_3 \ge 5$$

$$y_1 \ge 0; \quad y_2 \ge 0; \quad y_3 \ge 0$$

Example – 4 (Contd.)

Primal

Maximize $Z = 3x_1 + 5x_2$

s.t.

$$x_1 \le 4$$

$$2x_2 \le 12$$

$$3x_1 + 2x_2 \le 18$$

$$x_1 \ge 0; \quad x_2 \ge 0$$

Dual of a Dual is Primal

Dual

Minimize $Z' = 4y_1 + 12y_2 + 18y_3$

s.t.

$$y_1 + 0y_2 + 3y_3 \ge 3$$

$$0y_1 + 2y_2 + 2y_3 \ge 5$$

$$y_1 \ge 0; \quad y_2 \ge 0; \quad y_3 \ge 0$$

Example – 4 (Contd.)

The problem is converted to standard LP form

Maximize $Z'' = -Z' = -4y_1 - 12y_2 - 18y_3$

s.t. $y_1 + 3y_3 \geq 3 \longrightarrow y_1 + 3y_3 - y_4 = 3$

$2y_2 + 2y_3 \geq 5 \longrightarrow 2y_2 + 2y_3 - y_5 = 5$

$y_1 \geq 0; \quad y_2 \geq 0; \quad y_3 \geq 0$

$y_4 \geq 0; \quad y_5 \geq 0$

Example – 4 (Contd.)

Artificial variables

Maximize

$$Z'' = -4y_1 - 12y_2 - 18y_3 - MA_1 - MA_2$$

s.t.

$$y_1 + 3y_3 - y_4 + A_1 = 3$$

$$2y_2 + 2y_3 - y_5 + A_2 = 5$$

$$y_1 \geq 0; \quad y_2 \geq 0; \quad y_3 \geq 0$$

$$y_4 \geq 0; \quad y_5 \geq 0; \quad A_1 \geq 0; \quad A_2 \geq 0$$

Example – 4 (Contd.)

Transformation of Z'' row

$$Z'' + 4y_1 + 12y_2 + 18y_3 + MA_1 + MA_2 = 0$$

$$y_1 + 3y_3 - y_4 + A_1 = 3$$

$$2y_2 + 2y_3 - y_5 + A_2 = 5$$

y_1	y_2	y_3	y_4	y_5	A_1	A_2	b_i
4	12	18	0	0	M	M	0
1	0	3	-1	0	1	0	3
4 – M	12	18 – 3M	M	0	0	M	-3M
0	2	2	0	-1	0	1	5
4 – M	12 – 2M	18 – 5M	M	M	0	0	-8M

Example – 4 (Contd.)

Iteration – 1

Entering variable: y_3

Basis	Z''	y_1	y_2	y_3	y_4	y_5	A_1	A_2	b_i	b_i/a_{ij}
Z''	1	4 – M	12 – 2M	18 – 5M	M	M	0	0	-8M	–
A_1	0	1	0	3	-1	0	1	0	3	1
A_2	0	0	2	2	0	-1	0	1	5	2.5

Departing variable: A_1

Example – 4 (Contd.)

Iteration – 2

Entering variable: y_2

Basis	Z''	y_1	y_2	y_3	y_4	y_5	A_1	A_2	b_i	b_i/a_{ij}
Z''	1	$\frac{2M}{3}-2$	12 – 2M	0	$-\frac{2M}{3}+6$	M	$\frac{5M}{3}-6$	0	-3M-18M	–
y_3	0	1/3	0	1	-1/3	0	1/3	0	1	–
A_2	0	-2/3	2	0	2/3	-1	-2/3	1	3	3/2

Departing variable: A_2

Example – 4 (Contd.)

Iteration – 3 (Optimal Soln Dual)

Basis	Z''	y_1	y_2	y_3	y_4	y_5	A_1	A_2	b_i
Z''	1	2	0	0	2	6	M-2	M-6	-36
y_3	0	1/3	0	1	-1/3	0	1/3	0	1
y_2	0	-1/3	0	0	1/3	-1/2	-1/3	1/2	3/2

Example – 4 (Contd.)

Since all coefficients in the Z-row are non-negative this is the optimal solution.

$$Z'' = -36 \qquad Z' = -Z'' = 36$$

Same as the primal OF optimal value

$$y_1 = 0$$

$$y_2 = 3/2$$

$$y_3 = 1$$

Minimize $Z' = 4y_1 + 12y_2 + 18y_3$

s.t.

$$y_1 + 0y_2 + 3y_3 \geq 3$$

$$0y_1 + 2y_2 + 2y_3 \geq 5$$

$$y_1 \geq 0; \quad y_2 \geq 0; \quad y_3 \geq 0$$

Example – 4 (Contd.)

Iteration-3 of primal problem (Lecture-10 of the course):

Basis	Row	Z	x_1	x_2	x_3	x_4	x_5	b_i
Z	0	1	0	0	0	3/2	1	36
x_3	1	0	0	0	1	1/3	-1/3	2
x_2	2	0	0	1	0	1/2	0	6
x_1	3	0	1	0	0	-1/3	1/3	2

Example – 4 (Contd.)

$y_1 = 0$

$y_2 = 3/2$

$y_3 = 1$

Same as coefficients of slack variable in Z-row of final table of primal solution

Coefficients of y_4 and y_5 in Z'' row : (2, 6)

Correspond to primal solution, $x_1 = 2$

$x_2 = 6$

LP – Dual Problem

Primal and Dual problems relationship:

- If one problem (primal or dual) has an optimal feasible solution, the other problem also has an optimal feasible solution. The optimal value of the OF is the same for both problems.
- When one problem has no feasible solution (infeasible), the other problem is either infeasible or is unbounded.
- If one problem is unbounded, the other problem is infeasible.
- The final simplex table of one problem contains the complete solution of the other problem.

LP – Sensitivity

Sensitivity of Z:

- Dual variables indicate a change in Z value for a small change in R.H.S (b_i) of the particular constraint.

$$\Delta Z = y_i \times \Delta b_i$$

Change in the RHS of ith constraint

Dual variable associated with the ith constraint

For example,

the R.H.S of constraint (3) increased from 18 to 19,

$$\Delta b_i = 19 - 18 = 1$$

y_3 from solution of dual = 1.0

Increase in Z is,

$$\Delta Z = 1 \times 1 = 1$$

LP – Sensitivity

new value of Z is,

New value = Old value + ΔZ

= $36 + 1 = 37$

Dual variable, y_1, of constraint (1) is zero …. Z is insensitive to small changes in b_1 .

LP – Sensitivity

Sensitivity of optimal solution to:

- Change in coefficients c_j in the objective function.
- Addition of new variables.
- Change in constraint coefficients a_{ij}.
- Addition of new constraints.

$$Max \quad Z = \sum_{j=1}^{n} c_j x_j$$

$$\sum_{j=1}^{n} a_{ij} x_j = b_i$$

$$i = 1, 2, \ldots m$$

$$x_j \geq 0$$

$$j = 1, 2, \ldots n$$