

Water Resources Systems: Modeling Techniques and Analysis

Lecture - 13

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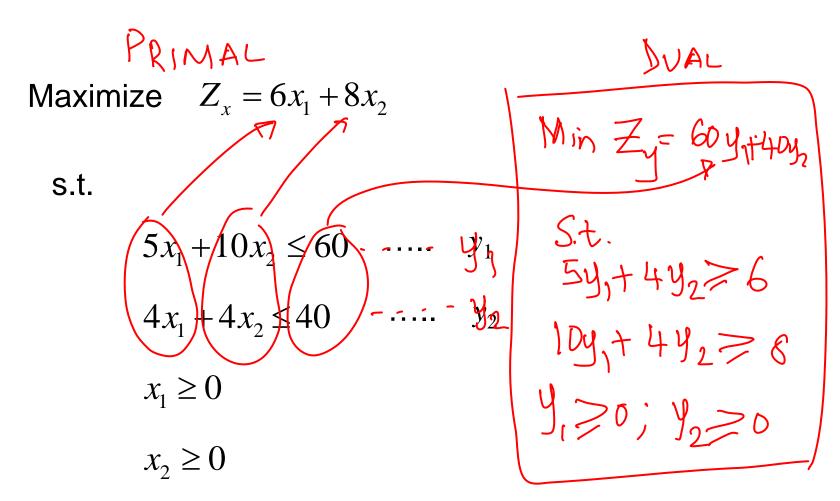
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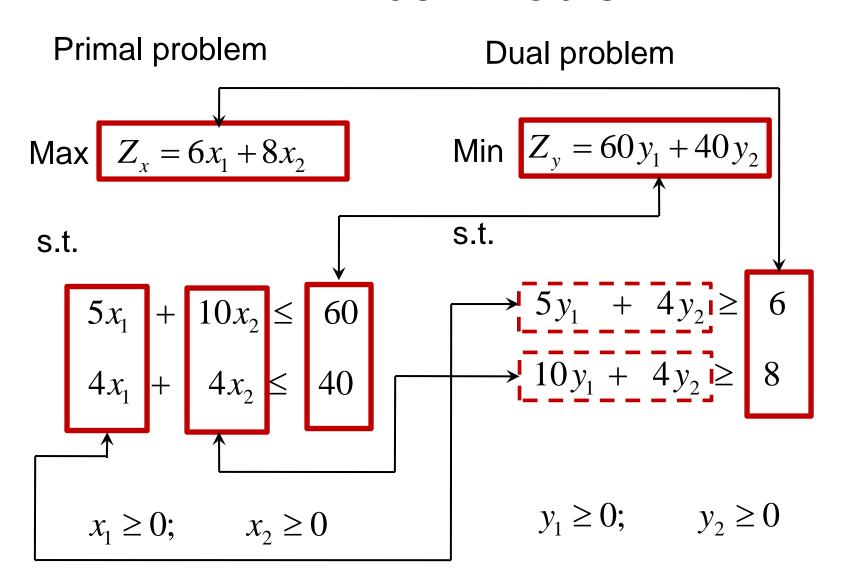
Summary of the previous lecture

Simplex Algorithm: :

- Example on use of artificial variables
- Unbounded solution
 - No departing variable can be found at some iteration
- Infeasible solution
 - One or more artificial variables remain in the basis even when optimality criterion is satisfied

LP - DUAL PROBLEM





- Formulation of dual from primal:
 - Express the primal as

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Maximization problem with ≤ or = constraints
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or

Minimization problem with \geq or = constraints

$\frac{\text{Primal}}{\text{OF: Maximize}} \qquad \qquad \frac{\text{Dual}}{\text{Minimize}}$ $\text{Variable } i \qquad \qquad \text{Constraint } i$ $j^{th} \text{ constraint} \qquad \qquad j^{th} \text{ variable}$

$$x_i \ge 0$$
 i^{th} constraint x_i unrestricted i^{th} constraint

$$j^{th}$$
 constraint ' \leq ' j^{th} variable ' \geq 0' j^{th} constraint ' $=$ ' j^{th} variable unrestricted

Primal

Dual

RHS of constraints (b_i)

Coefficients in objective function

Coefficients in objective function

RHS of constraints

e.g.,

Maximize
$$Z = 2x_1 + x_2$$
 s.t.

$$x_1 - 2x_2 \ge 2$$

$$x_1 + 2x_2 = 8$$

$$x_1 - x_2 \le 11$$

$$x_1 \ge 0$$

 x_2 unrestricted

The problem is rewritten as Minimize $Z'=-2x_1-x_2$ s.t. $x_1-2x_2\geq 2$ y_1 $x_1+2x_2=8$ y_2 $-x_1+x_2\geq -11$ y_3 $x_1\geq 0; x_2$ unrestricted

Dual of the problem

Maximize
$$2y_1 + 8y_2 - 11y_3$$

s.t.

$$1 \times y_1 + 1 \times y_2 - 1 \times y_3 \le -2$$

$$-2 \times y_1 + 2 \times y_2 + \cancel{\triangleright} \times y_3 = -1$$

 $y_1 \ge 0$; y_2 unrestricted

$$y_3 \ge 0$$

Because second constraint of primal is '='

$$Z' = -2x_1 - x_2$$
 $x_1 - 2x_2 \ge 2 \cdots y_1$
 $x_1 + 2x_2 = 8 \cdots y_2$
 $-x_1 + x_2 \ge -11 \cdots y_3$
 $x_1 \ge 0; x_2 \text{ unrestricted}$

Because x_2 is unrestricted

Example – 4

Solve the dual of the problem Maximize

$$Z = 3x_1 + 5x_2$$

s.t.

$$x_{1} \le 4$$

$$2x_{2} \le 12$$

$$3x_{1} + 2x_{2} \le 18$$

$$x_{1} \ge 0$$

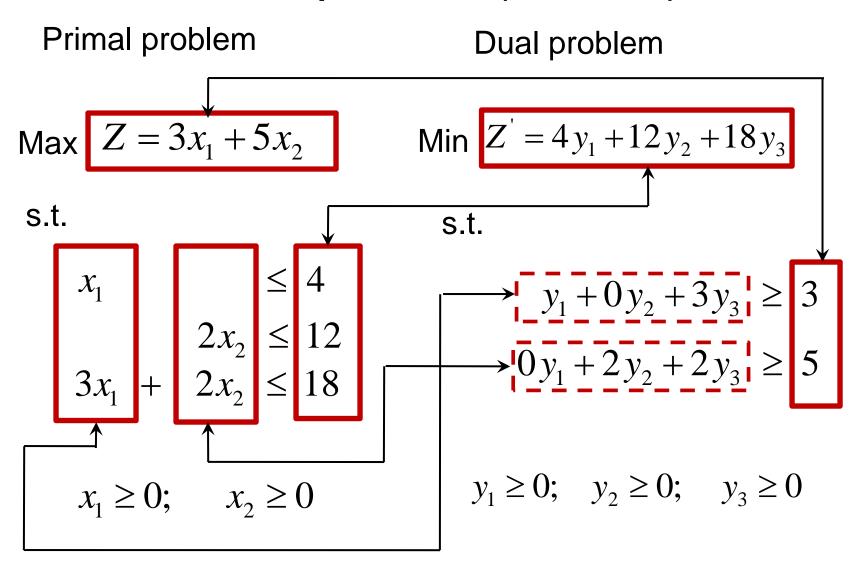
$$x_{2} \ge 0$$

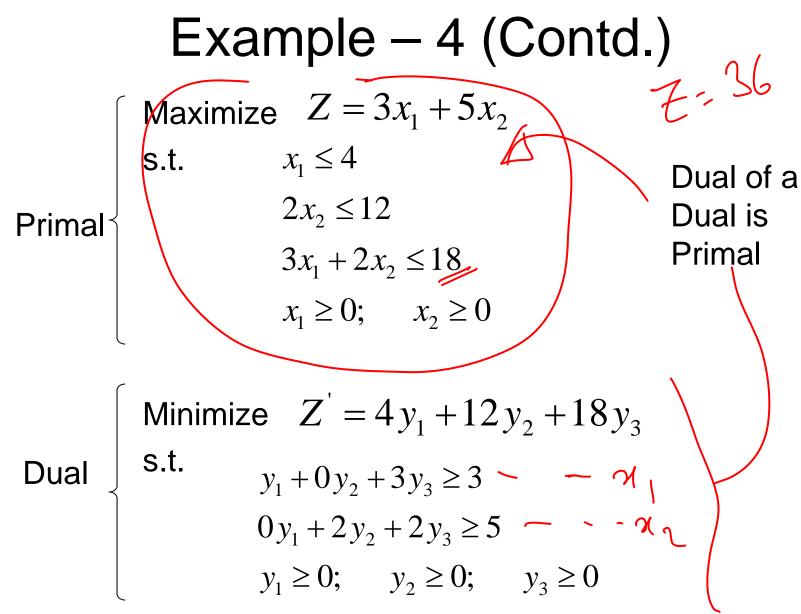
Primal problem

Maximize
$$Z = 3x_1 + 5x_2$$

s.t.

$$x_1 \le 4$$
 y_1
 $2x_2 \le 12$ y_2
 $3x_1 + 2x_2 \le 18$ y_3
 $x_1 \ge 0$; $x_2 \ge 0$





The problem is converted to standard LP form

Maximize
$$Z'' = -Z' = -4y_1 - 12y_2 - 18y_3$$

s.t.
$$y_1 + 3y_3 \ge 3 \longrightarrow y_1 + 3y_3 - y_4 = 3$$

 $2y_2 + 2y_3 \ge 5 \longrightarrow 2y_2 + 2y_3 - y_5 = 5$
 $y_1 \ge 0; \quad y_2 \ge 0; \quad y_3 \ge 0$
 $y_4 \ge 0; \quad y_5 \ge 0$

Artificial variables

Maximize

$$Z'' = -4y_1 - 12y_2 - 18y_3 - MA_1 - MA_2$$

s.t.

$$y_1 + 3y_3 - y_4 + A_1 = 3$$

$$2y_2 + 2y_3 - y_5 + A_2 = 5$$

$$y_1 \ge 0; \quad y_2 \ge 0; \quad y_3 \ge 0$$

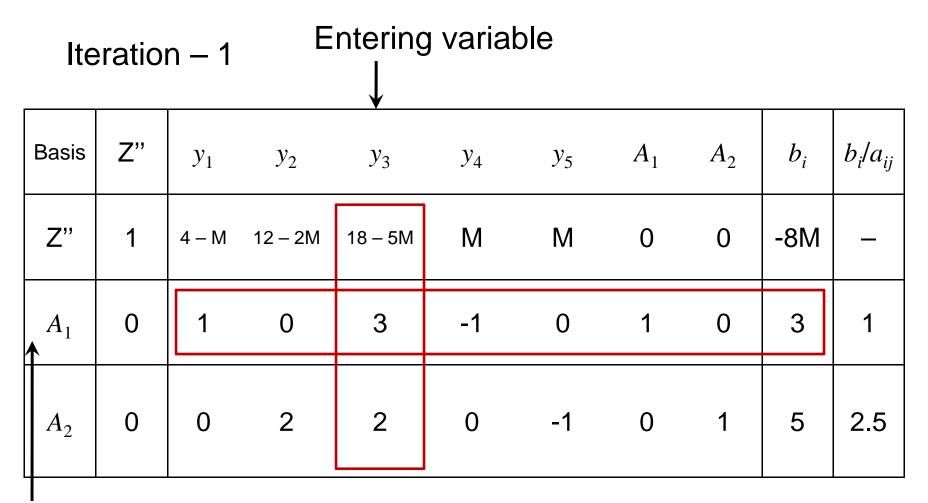
$$y_4 \ge 0$$
; $y_5 \ge 0$; $A_1 \ge 0$; $A_2 \ge 0$

Transformation of Z" row

$$Z'' + 4y_1 + 12y_2 + 18y_3 + MA_1 + MA_2 = 0$$
$$y_1 + 3y_3 - y_4 + A_1 = 3$$

$$2y_2 + 2y_3 - y_5 + A_2 = 5$$

y_1	y_2	y_3	y_4	y_5	A_1	A_2	b_i
4	12	18	0	0	М	M	0
1	0	3	-1	0	1	0	3
4 – M	12	18 – 3M	M	0	0	М	-3M
0	2	2	0	-1	0	1	5
4 – M	12 – 2M	18 – 5M	М	М	0	0	-8M



Departing variable

Entering variable Iteration – 2 **Z**" b_i/a_{ii} Basis y_2 y_3 y_4 y_5 A_1 A_2 b_i y_1 $-\frac{2M}{3}+6$ M $\frac{5M}{3}-6$ 0 $\frac{2M}{3}$ - 2 12 - 2M -3M-0 1 18M 1/3 1/3 -1/3 0 0 0 0 1 y_3 0 -2/3 2 2/3 -1 -2/3 3/2 0 1

Departing variable

Iteration – 3

Cophnal Sola Dual)

Basis	Z"	y_1	y_2	y_3	y_4	y_5	A_1	A_2	b_i
Z"	1	2	0	0	2	6	M-2	M-6	-36
<i>y</i> ₃	0	1/3	0	1	-1/3	0	1/3	0	1
y_2	0	-1/3	0	0	1/3	-1/2	-1/3	1/2	3/2

Since all coefficients in the Z-row are non-negative this is the optimal solution.

$$Z'' = -36$$

$$Z' = -Z'' = 36$$
 Same as the primal OF optimal value

$$y_1 = 0$$

$$y_1 = 0$$

$$y_2 = \frac{3}{2}$$

$$y_3 = 1$$

Minimize
$$Z' = 4y_1 + 12y_2 + 18y_3$$

s.t. $y_1 + 0y_2 + 3y_3 > 3$

$$y_1 + 0y_2 + 3y_3 \ge 3$$

 $0y_1 + 2y_2 + 2y_3 \ge 5$
 $y_1 \ge 0; \quad y_2 \ge 0; \quad y_3 \ge 0$

Iteration-3 of primal problem (Lecture-10 of the course):

Basis	Row	Z	x_1	x_2	x_3	x_4	x_5	b_i
Z	0	1	0	0	0	3/2	1	36
x_3	1	0	0	0	1	1/3	-1/3	2
x_2	2	0	0	1	0	1/2	0	6
x_1	3	0	1	0	0	-1/3	1/3	2

$$y_1 = 0$$

$$y_2 = \frac{3}{2}$$

$$y_3 = 1$$

Same as coefficients of slack variable in Z-row of final table of primal solution

Coefficients of y_4 and y_5 in Z" row : (2, 6)

Correspond to primal solution, $x_1 = 2$

$$x_2 = 6$$

Primal and Dual problems relationship:

- If one problem (primal or dual) has an optimal feasible solution, the other problem also has an optimal feasible solution. The optimal value of the OF is the same for both problems.
- When one problem has no feasible solution (infeasible), the other problem is either infeasible or is unbounded.
- If one problem is unbounded, the other problem is infeasible.
- The final simplex table of one problem contains the complete solution of the other problem.

LP – Sensitivity

Sensitivity of Z:

Dual variables indicate a change in Z value for a small change in R.H.S (b_i) of the particular constraint. $\Delta Z = \underbrace{y_i \times \Delta b_i}_{i} \qquad \text{the constant}$ e, but variable_{i}

$$\Delta Z = y_i \times \Delta b_i$$

For example,

the R.H.S of constraint (3) increased from 18 to 19, associated work

$$\Delta b_i = 19 - 18 = 1$$
 y_3 from solution of dual = 1.0

Increase in Z is,

$$\Delta Z = 1 \times 1 = 1$$

LP – Sensitivity

new value of Z is, New value = Old value + ΔZ = 36 + 1 = 37

Dual variable, y_1 , of constraint (1) is zero Z is insensitive to small changes in b_1 .

LP – Sensitivity

Sensitivity of optimal solution to:

- Change in coefficients c_j in the objective function.
- Addition of new variables.
- Change in constraint coefficients a_{ii} .
- Addition of new constraints.

$$Max Z = \sum_{j=1}^{n} c_j x_j$$

$$\sum_{j=1}^{n} a_{ij} x_j = b_i$$

$$i = 1, 2, \dots m$$

$$x_j \ge 0$$

$$j = 1, 2, \dots n$$