INDIAN INSTITUTE OF SCIENCE

Water Resources Systems:

Modeling Techniques and Analysis

Lecture - 30

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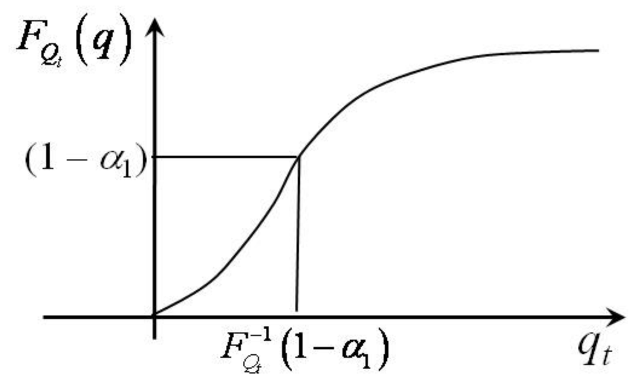
Summary of the previous lecture

- Chance constrained LP for reservoir operation and design.

Chance constraint $P\left[R_t \geq D_t\right] \geq \alpha_1$



Linear Decision Rule (LDR):

$$R_t = S_t + Q_t - b_t$$

Deterministic equivalent of the chance constraint

$$\left(D_t + b_t - b_{t-1}\right) \leq F_{Q_t}^{-1}\left(1-\alpha_1\right)$$

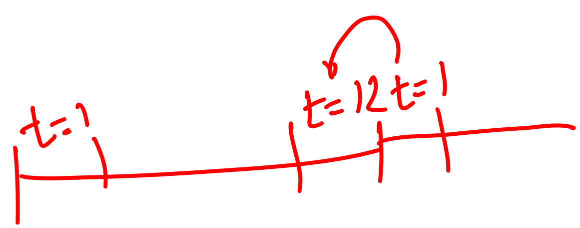
Chance Constrained LP

The complete deterministic equivalent of CCLP is written as

b_0 @ $t=1$
$= b_{12}$

$$
\begin{aligned}
&\text{Min} \quad K \\
&\text{s.t.} \quad \left.
\begin{aligned}
&\left(D_t + b_t - b_{t-1}\right) \le F_{Q_t}^{-1}\left(1-\alpha_1\right) \\
&\left(R_t^{\max} + b_t - b_{t-1}\right) \ge F_{Q_t}^{-1}\left(\alpha_2\right) \\
&b_{t-1} \le K \\
&b_{t-1} \ge S_{\min} \\
&b_t \ge 0
\end{aligned}
\right\} \forall\, t \\
&K \ge 0
\end{aligned}
$$

$$
\boxed{
\begin{aligned}
&\text{Min} \quad K \\
&\text{s.t. } P\left[R_t \ge D_t\right] \ge \alpha_1 \\
&\quad P\left[R_t \le R_t^{\max}\right] \ge \alpha_2 \\
&\quad P\left[S_t \le K\right] \ge \alpha_3 \\
&\quad P\left[S_t \ge S_{\min}\right] \ge \alpha_4
\end{aligned}
}
$$



Chance Constrained LP

- Set $b_0 = b_{12}$ for a steady state solution (12 months period)
- Depending on the nature of LDR used, the decision parameters, b_t, may be unrestricted in sign

For example, in the LDR

$$R_t = S_t - b_t$$

the decision parameter, b_t, is allowed to take –ve values.

When b_t is –ve, the release is more than the initial storage in period t, and the volume of release over the available storage, S_t, is provided by part of the inflow, Q_t, not included in LDR

Example – 1

With the LDR, $R_t = S_t + (1 - \zeta)Q_t - b_t$ where $0 \le \zeta \le 1$,
Obtain the deterministic equivalent of the storage chance constraint , $P[S_t \le K] \ge 0.8$. Assume that the flows, Q_t, follow exponential distribution with pdf given by,

$$f(q) = \beta e^{-\beta q} \qquad q > 0$$

$f(q) = \beta e^{-\beta q}$

Exponential distribution

$t = 1, \beta = 2$

$t = 2, \beta = 5$

$t = 3, \beta = 7$

K is the known reservoir capacity. Neglect evaporation losses.

Example – 1 (Contd.)

Continuity

LDR, $R_t = S_t + (1 - \zeta)Q_t - b_t$

$$S_{t+1} = S_t + Q_t - R_t$$ ….. neglecting evaporation losses.

$$= S_t + Q_t - \left\{ S_t + \left(1 - \zeta\right) Q_t - b_t \right\}$$

$$= \zeta Q_t + b_t$$

$$S_t = \zeta Q_{t-1} + b_{t-1}$$

Example – 1 (Contd.)

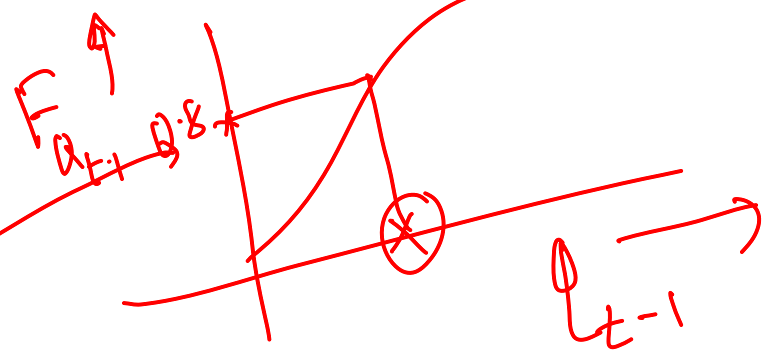
Deterministic equivalent of $P[S_t \leq K] \geq 0.8$

$$P\left[\zeta Q_{t-1} + b_{t-1} \leq K\right] \geq 0.8$$

$$P\left[\zeta Q_{t-1} \leq K - b_{t-1}\right] \geq 0.8$$

$F(x) = P[X \leq x]$

$$P\left[Q_{t-1} \leq \frac{K - b_{t-1}}{\zeta}\right] \geq 0.8$$



$$F_{Q_{t-1}}\left(\frac{K - b_{t-1}}{\zeta}\right) \geq 0.8$$

$$\frac{K - b_{t-1}}{\zeta} \geq F_{Q_{t-1}}^{-1}(0.8)$$

Example – 1 (Contd.)

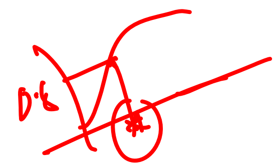
Q_t follows exponential distribution

$$f(q) = \beta e^{-\beta q} \qquad q > 0$$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

CDF is

$$F(q) = 1 - e^{-\beta q} \qquad q > 0$$



For $t = 1 \quad \Rightarrow \qquad t - 1 = 3$

$$F(Q_3)$$

$$\frac{K - b_3}{\zeta} \geq F_{Q_3}^{-1}(0.8)$$

Set $b_0 = b_3$

From the distribution,

$$1 - e^{-7 q_3} = 0.8$$

$\beta = 7$ for period $t = 3$

$$q_3 = 0.23$$

Example – 1 (Contd.)

For $t = 2$

$$\frac{K - b_1}{\zeta} \geq F_{Q_1}^{-1}(0.8)$$

From the distribution, $1 - e^{-3q_1} = 0.8$

$\beta = 2$ for period $t = 1$

$$q_1 = 0.536$$

For $t = 3$

$$\frac{K - b_2}{\zeta} \geq F_{Q_2}^{-1}(0.8)$$

From the distribution, $1 - e^{-5q_2} = 0.8$

$\beta = 5$ for period $t = 2$

$$q_2 = 0.322$$

Example – 1 (Contd.)

Thus, the deterministic equivalents are

$$\frac{K-b_3}{\zeta} \geq 0.23 \qquad \ldots. \quad t=1$$

$$\frac{K-b_1}{\zeta} \geq 0.536 \qquad \ldots. \quad t=2$$

$$\frac{K-b_2}{\zeta} \geq 0.322 \qquad \ldots. \quad t=3$$

$P[S_t \leq K] \geq 0.8 \quad \forall t$

Known

Example – 2

Write down the complete deterministic equivalent of the following three-period, chance constrained LP problem.

Minimize K

s.t $P[S_{\min} \le S_t \le K] \ge 0.9 \qquad \forall t$

$P[R_t \le R^t_{\max}] \ge 0.95 \qquad \forall t$

$P[R_t \ge D_t] \ge 0.75 \qquad \forall t$

CCLP

$R^t_{\min}$

Min. Release [Demand]

Use linear decision rule, $R_t = S_t - b_t$

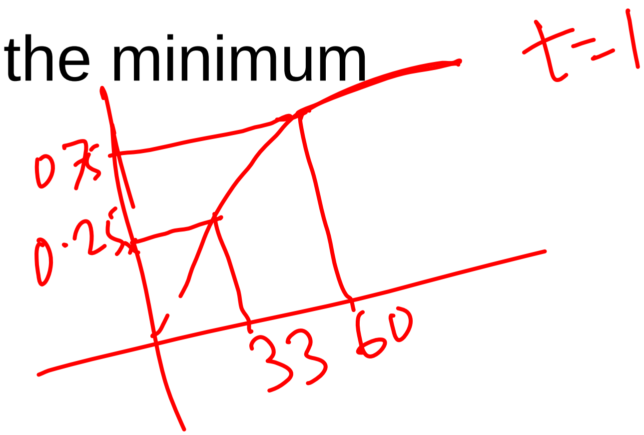
Example – 2 (Contd.)

Inflow CDF values and R_{max}, R_{min} and S_{min} values:

Δt

t	$F^{-1}(0.0)$	$F^{-1}(0.1)$	$F^{-1}(0.25)$	$F^{-1}(0.75)$	$F^{-1}(0.9)$	$F^{-1}(0.95)$	R_{max}	R_{min}	S_{min}
1	0	12	33	60	90	93	90	24	2
2	0	3	20	48	60	80	84	20	2
3	0	6	21	36	72	85	84	20	2

Solve the CCLP problem to obtain the minimum capacity required.



Example – 2 (Contd.)

Solution:

Continuity

LDR is $$R_t = S_t - b_t$$

$$S_{t+1} = S_t + Q_t - R_t$$

$$= S_t + Q_t - S_t + b_t$$

$$= Q_t + b_t$$

$$S_t = Q_{t-1} + b_{t-1}$$

Example – 2 (Contd.)

Deterministic equivalent of

$$P[S_{\min} \le S_t \le K] \ge 0.9$$

The constraint can be written as

$$P[S_{\min} \le S_t] \ge 0.9$$

$$P[S_t \le K] \ge 0.9$$

Known

Deterministic equivalent of

$$P[S_{\min} \le S_t] \ge 0.9$$

$$P[S_{\min} \le Q_{t-1} + b_{t-1}] \ge 0.9$$

$$P\,[Q_{t-1} + b_{t-1} \ge S_{\min}] \ge 0.9$$

Example – 2 (Contd.)

$$P[Q_{t-1} \geq S_{\min} - b_{t-1}] \geq 0.9$$

$$P[Q_{t-1} \leq S_{\min} - b_{t-1}] \leq (1 - 0.9)$$

$$P[Q_{t-1} \leq 2 - b_{t-1}] \leq 0.1$$

(as S_{min}=2)

$$F_{Q_{t-1}}(2 - b_{t-1}) \leq 0.1$$

$$2 - b_{t-1} \leq F^{-1}_{Q_{t-1}}(0.1)$$

$t = 1, 2, 3$

$$2 - b_3 \leq F^{-1}_{Q_3}(0.1) \longrightarrow 2 - b_3 \leq 6$$ ...for t=1

$$2 - b_1 \leq F^{-1}_{Q_1}(0.1) \longrightarrow 2 - b_1 \leq 12$$ ...for t=2

$$2 - b_2 \leq F^{-1}_{Q_2}(0.1) \longrightarrow 2 - b_2 \leq 3$$ ...for t=3

Example – 2 (Contd.)

Deterministic equivalent of $P[S_t \leq K] \geq 0.9$

$K - b_3 \geq 72$ ...for t=1

$K - b_1 \geq 90$ ...for t=2

$K - b_2 \geq 60$ ...for t=3

Deterministic equivalent of $P[R_t \leq R_{\max}^t] \geq 0.95$

$90 + b_1 - b_3 \geq 85$ ...for t=1

$84 + b_2 - b_1 \geq 93$ ...for t=2

$84 + b_3 - b_2 \geq 80$ ...for t=3

Example – 2 (Contd.)

Deterministic equivalent of $P[R_t \geq D_t] \geq 0.75$

$24 + b_1 - b_3 \leq 21$ ...for t=1

$20 + b_2 - b_1 \leq 33$ ...for t=2

$20 + b_3 - b_2 \leq 20$ ...for t=3

Example – 2 (Contd.)

Thus, the deterministic equivalent optimization model is

Minimize K

s.t

$$2 - b_3 \le 6$$
$$2 - b_1 \le 12$$
$$2 - b_2 \le 3$$
$$K - b_3 \ge 72$$
$$K - b_1 \ge 90$$
$$K - b_2 \ge 60$$
$$90 + b_1 - b_3 \ge 85$$
$$84 + b_2 - b_1 \ge 93$$
$$84 + b_3 - b_2 \ge 80$$

$$24 + b_1 - b_3 \le 21$$
$$20 + b_2 - b_1 \le 33$$
$$20 + b_3 - b_2 \le 20$$

The solution of this model results in,
$K = 90$; $b_1 = 0$; $b_2 = 9$; $b_3 = 5$