INDIAN INSTITUTE OF SCIENCE

# Water Resources Systems:

## Modeling Techniques and Analysis

Lecture - 33

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# Summary of the previous lecture

- Stochastic dynamic programming

Steady state policy:

$$f_t^n(k,i) = \underset{\{feasible\ l\}}{Max}\left[B_{kil\,t} + \sum_j P_{ij}^t f_{t+1}^{n-1}(l,j)\right]$$

$$\underbrace{f_t^{n+T}(k,i) - f_t^n(k,i)}$$

Expected annual performance

Remains constant $\forall\ k$, $i$ and $t$ .

# Example – 1

Obtain the steady state policy with an objective to minimize the expected value of the sum of the square of deviations of release and storage from their respective targets, over a year with two periods. Neglect the evaporation loss. If the release is greater than release target, the deviation is set to zero. The data is as follows.

Period $t = 1$

| $i$ | $Q_i^t$ | $k$ | $S_k^t$ |
|---|---|---|---|
| 1 | 15 | 1 | 30 |
| 2 | 25 | 2 | 40 |

Period $t = 2$

| $i$ | $Q_i^t$ | $k$ | $S_k^t$ |
|---|---|---|---|
| 1 | 35 | 1 | 20 |
| 2 | 45 | 2 | 30 |

# Example – 1 (Contd.)

Target storage $T_s$ = 30
Target release $T_r$ = 30

Inflow transition probabilities:

| | $t = 2$ | | |
|---|---|---|---|
| | | $j$ | |
| $t = 1$ | $i$ | 1 | 2 |
| | 1 | 0.5 | 0.5 |
| | 2 | 0.3 | 0.7 |

| | $t = 1$ | | |
|---|---|---|---|
| | | $j$ | |
| $t = 2$ | $i$ | 1 | 2 |
| | 1 | 0.4 | 0.6 |
| | 2 | 0.8 | 0.2 |

# Example – 1 (Contd.)

Solution:

The system performance measure, $B_{kilt}$, is the sum of the square of deviations of release and storage from their respective targets

$$B_{kilt} = \left(R_{kilt} - T_r\right)^2 + \left(S_k^t - T_s\right)^2$$

Target storage $T_s$ = 30
Target release $T_r$ = 30

The system performance measure, $B_{kilt}$, is tabulated $\forall$ $k, i, l$ and $t$.

# Example – 1 (Contd.)

Period $t = 1$

| $k$ | $S_k^t$ | $i$ | $Q_i^t$ | $l$ | $S_l^{t+1}$ | $R_{kilt}$ | $(S_k^t\text{-}T_s)^2$ | $(R_{kilt}\text{-}T_r)^2$ | $B_{kilt}$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 30 | 1 | 15 | 1 | 20 | 25 | 0 | 25 | 25 |
| 1 | 30 | 1 | 15 | 2 | 30 | 15 | 0 | 225 | 225 |
| 1 | 30 | 2 | 25 | 1 | 20 | 35 | 0 | 0 | 0 |
| 1 | 30 | 2 | 25 | 2 | 30 | 25 | 0 | 25 | 25 |
| 2 | 40 | 1 | 15 | 1 | 20 | 35 | 100 | 0 | 100 |
| 2 | 40 | 1 | 15 | 2 | 30 | 25 | 100 | 25 | 125 |
| 2 | 40 | 2 | 25 | 1 | 20 | 45 | 100 | 0 | 100 |
| 2 | 40 | 2 | 25 | 2 | 30 | 35 | 100 | 0 | 100 |

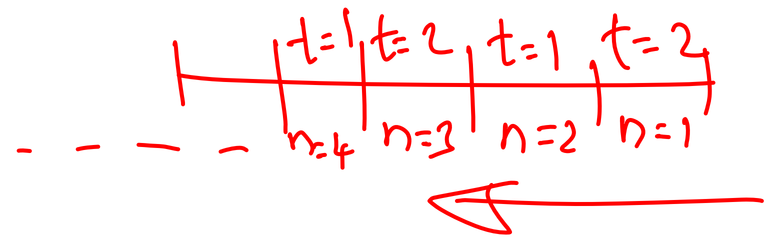
# Example – 1 (Contd.)

Period $t = 2$

| $k$ | $S_k^t$ | $i$ | $Q_i^t$ | $l$ | $S_l^{t+1}$ | $R_{kilt}$ | $(S_k^t - T_s)^2$ | $(R_{kilt} - T_r)^2$ | $B_{kilt}$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 20 | 1 | 35 | 1 | 30 | 25 | 100 | 25 | 125 |
| 1 | 20 | 1 | 35 | 2 | 40 | 15 | 100 | 225 | 325 |
| 1 | 20 | 2 | 45 | 1 | 30 | 35 | 100 | 0 | 100 |
| 1 | 20 | 2 | 45 | 2 | 40 | 25 | 100 | 25 | 125 |
| 2 | 30 | 1 | 35 | 1 | 30 | 35 | 0 | 0 | 0 |
| 2 | 30 | 1 | 35 | 2 | 40 | 25 | 0 | 25 | 25 |
| 2 | 30 | 2 | 45 | 1 | 30 | 45 | 0 | 0 | 0 |
| 2 | 30 | 2 | 45 | 2 | 40 | 35 | 0 | 0 | 0 |

# Example – 1 (Contd.)

$n = 1 \quad t = 2$




$$f_1^2(k,i) = \underset{\{feasible\ l\}\ j}{Min} \left[ B_{kilt} \right] \qquad \forall k,i$$

| $k$ | $i$ | $B_{kilt}$ | | $f_1^2(k,i)$ | $l*$ |
|---|---|---|---|---|---|
| | | $l = 1$ | $l = 2$ | | |
| 1 | 1 | 125.00 | 325.00 | 125.00 | 1 |
| 1 | 2 | 100.00 | 125.00 | 100.00 | 1 |
| 2 | 1 | 0.00 | 25.00 | 0.00 | 1 |
| 2 | 2 | 0.00 | 0.00 | 0.00 | 1,2 |

# Example – 1 (Contd.)

$n = 2 \quad t = 1$

$$f_2^1(k,i) = \underset{\{feasible\ l\}}{Min} \left[ B_{kilt} + \sum_j P_{ij}^1 f_1^1(l,j) \right] \qquad \forall\ k,i$$

$k = 1, \ i = 1, l = 1;$

$$B_{kil1} + \sum P_{ij}^1 f_1^1(l,j) = 25.0 + 0.5 \times 125.0 + 0.5 \times 100.0$$

$$= 137.5$$

$k = 1, \ i = 1, l = 2;$

$$B_{kil1} + \sum P_{ij}^1 f_1^1(l,j) = 225.0 + 0.5 \times 0.0 + 0.5 \times 0.0$$

$$= 225.0$$

# Example – 1 (Contd.)

$n = 2 \quad t = 1$

$k = 1, \; i = 2, l = 1;$

$$B_{kil\ 1} + \sum P_{ij}^{1} f_{1}^{1}(l, j) = 0.0 + 0.3 \times 125.0 + 0.7 \times 100.0$$

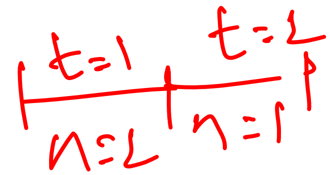
$$= 107.5$$

$k = 1, \; i = 2, l = 2;$

$$B_{kil\ 1} + \sum P_{ij}^{1} f_{1}^{1}(l, j) = 25.0 + 0.3 \times 0.0 + 0.7 \times 0.0$$

$$= 25.0$$

# Example – 1 (Contd.)




$n = 2 \quad t = 1$

| $k$ | $i$ | $B_{kil1} + \sum P_{ij}^{1} f_{1}^{1}(l, j)$ | | $f_{2}^{1}(k, i)$ | $l^*$ |
|---|---|---|---|---|---|
| | | $l = 1$ | $l = 2$ | | |
| 1 | 1 | 137.50 | 225.00 | 137.50 | 1 |
| 1 | 2 | 107.50 | 25.00 | 25.00 | 2 |
| 2 | 1 | 212.50 | 125.00 | 125.00 | 2 |
| 2 | 2 | 207.50 | 100.00 | 100.00 | 2 |

# Example – 1 (Contd.)

$n = 3 \;\; t = 2$

$$f_3^2(k,i) = \underset{\{feasible\ l\}}{Min} \left[ B_{kil2} + \sum_j P_{ij}^1 f_2^1(l, j) \right] \qquad \forall\ k,i$$

| $k$ | $i$ | $B_{kil2} + \sum P_{ij}^1 f_2^1(l, j)$ | | $f_3^2(k,i)$ | $l^*$ |
|---|---|---|---|---|---|
| | | $l = 1$ | $l = 2$ | | |
| 1 | 1 | 195.00 | 435.00 | 195.00 | 1 |
| 1 | 2 | 215.00 | 245.00 | 215.00 | 1 |
| 2 | 1 | 70.00 | 135.00 | 70.00 | 1 |
| 2 | 2 | 115.00 | 120.00 | 115.00 | 1 |

# Example – 1 (Contd.)

$n = 4 \quad t = 1$

$$f_4^1(k,i) = \underset{\{feasible\ l\}}{Min}\left[ B_{kil1} + \sum_j P_{ij}^2 f_3^2(l,j) \right] \qquad \forall\ k,i$$

| $k$ | $i$ | $B_{kil1} + \sum P_{ij}^2 f_3^2(l,j)$ | | $f_4^1(k,i)$ | $l^*$ |
|---|---|---|---|---|---|
| | | $l = 1$ | $l = 2$ | | |
| 1 | 1 | 230.00 | 317.50 | 230.00 | 1 |
| 1 | 2 | 209.00 | 126.50 | 126.50 | 2 |
| 2 | 1 | 305.00 | 217.50 | 217.50 | 2 |
| 2 | 2 | 309.00 | 201.50 | 201.50 | 2 |

# Example – 1 (Contd.)

$n = 5 \quad t = 1$

$$f_5^2(k,i) = \underset{\{feasible\ l\}}{Min} \left[ B_{kil2} + \sum_j P_{ij}^1 f_4^1(l,j) \right] \qquad \forall\ k,i$$

| $k$ | $i$ | $B_{kil2} + \sum P_{ij}^1 f_4^1(l,j)$ | | $f_5^2(k,i)$ | $l^*$ |
|---|---|---|---|---|---|
| | | $l = 1$ | $l = 2$ | | |
| 1 | 1 | 292.90 | 532.90 | 292.90 | 1 |
| 1 | 2 | 309.30 | 339.30 | 309.30 | 1 |
| 2 | 1 | 167.90 | 232.90 | 167.90 | 1 |
| 2 | 2 | 209.30 | 214.30 | 209.30 | 1 |

# Example – 1 (Contd.)

$n = 6 \quad t = 1$

$$f_6^1(k,i) = \underset{\{feasible\ l\}}{Min} \left[ B_{kil1} + \sum_j P_{ij}^2 f_5^2(l, j) \right] \qquad \forall\ k,i$$

| $k$ | $i$ | $B_{kil1} + \sum P_{ij}^2 f_5^2(l, j)$ | | $f_6^1(k,i)$ | $l^*$ |
|---|---|---|---|---|---|
| | | $l = 1$ | $l = 2$ | | |
| 1 | 1 | 326.10 | 413.60 | 326.10 | 1 |
| 1 | 2 | 304.38 | 221.88 | 221.88 | 2 |
| 2 | 1 | 401.10 | 313.60 | 313.60 | 2 |
| 2 | 2 | 404.38 | 296.88 | 296.88 | 2 |

# Example – 1 (Contd.)

$n = 7 \quad t = 2$

$$f_7^2(k,i) = \underset{\{feasible\ l\}}{Min}\left[B_{kil2} + \sum_j P_{ij}^1 f_6^1(l, j)\right] \qquad \forall\ k,i$$

| $k$ | $i$ | $B_{kil2} + \sum P_{ij}^1 f_6^1(l, j)$ | | $f_7^2(k,i)$ | $l^*$ |
|---|---|---|---|---|---|
| | | $l = 1$ | $l = 2$ | | |
| 1 | 1 | 388.57 | 628.57 | 388.57 | 1 |
| 1 | 2 | 405.26 | 435.26 | 405.26 | 1 |
| 2 | 1 | 263.57 | 328.57 | 263.57 | 1 |
| 2 | 2 | 305.26 | 310.26 | 305.26 | 1 |

# Example – 1 (Contd.)

$n = 8 \quad t = 1$

$$f_8^1(k,i) = \underset{\{feasible\ l\}}{Min} \left[ B_{kil1} + \sum_j P_{ij}^2 f_7^2(l, j) \right] \qquad \forall\ k,i$$

| $k$ | $i$ | $B_{kil1} + \sum P_{ij}^2 f_7^2(l, j)$ | | $f_8^1(k,i)$ | $l^*$ |
|---|---|---|---|---|---|
| | | $l = 1$ | $l = 2$ | | |
| 1 | 1 | 421.91 | 509.41 | 421.91 | 1 |
| 1 | 2 | 400.25 | 317.75 | 317.75 | 2 |
| 2 | 1 | 496.91 | 409.41 | 409.41 | 2 |
| 2 | 2 | 500.25 | 392.75 | 392.75 | 2 |

# Example – 1 (Contd.)

$n = 6 \quad t = 1$

| $f_6^1(k,i)$ | $l^*$ |
|---|---|
| 326.10 | 1 |
| 221.88 | 2 |
| 313.60 | 2 |
| 296.88 | 2 |

$n = 8 \quad t = 1$

| $f_8^1(k,i)$ | $l^*$ | k, i |
|---|---|---|
| 421.91 | 1 | (1,1) |
| 317.75 | 2 | (1,2) |
| 409.41 | 2 | (2,1) |
| 392.75 | 2 | (2,2) |

Steady state policy: $f_t^{n+T}(k,i) - f_t^{n}(k,i)$

$$f_8^1(1,1) - f_6^1(1,1) = 421.91 - 326.10 = 95.81$$

$$f_8^1(1,2) - f_6^1(1,2) = 317.75 - 221.88 = 95.87$$

approx. equal.

# Example – 1 (Contd.)

$n = 5 \;\; t = 2$

| $f_5^2(k,i)$ | $l^*$ |
|---|---|
| 292.90 | 1 |
| 309.30 | 1 |
| 167.90 | 1 |
| 209.30 | 1 |

$n = 7 \;\; t = 2$

| $f_7^2(k,i)$ | $l^*$ |
|---|---|
| 388.57 | 1 |
| 405.26 | 1 |
| 263.57 | 1 |
| 305.26 | 1 |

Steady state policy: $f_t^{n+T}(k,i) - f_t^n(k,i)$

$$f_7^2(1,1) - f_5^2(1,1) = 388.57 - 292.90 = 95.67$$

$$f_7^2(1,2) - f_5^2(1,2) = 405.26 - 309.30 = 95.96$$

# Example – 1 (Contd.)

Steady state policy:

Period $t = 1$

| $k$ | $i$ | $l^*$ |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 2 | 2 |
| 2 | 1 | 2 |
| 2 | 2 | 2 |

Period $t = 2$

| $k$ | $i$ | $l^*$ |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 2 | 1 |
| 2 | 1 | 1 |
| 2 | 2 | 1 |

# Stochastic Dynamic Programming

Steady state probabilities:

$l^*(k,i,t)$: Steady state policy for a given $k$ and $i$, in time period $t$.

For a unique $l^*(k,i,t)$ ,
steady state probabilities of $R_{kilt}$ may be written as
$PR_{kit}$ without the index $l$.

# Stochastic Dynamic Programming

Steady state probabilities:

$$PR_{ljt+1} = \sum_{k}\sum_{i} PR_{kit} P_{ij}^{t} \qquad \forall\, l, j \text{ and } t \quad \text{———— ①}$$

$$l = l^{*}(k,i,t)$$

NPTEL

This is a selective summation over only those initial storage and inflow indices $k$ and $i$ in period $t$ that result in the same $l = l^{*}(k,i,t)$

$$\sum_{k}\sum_{i} PR_{ki\,t} = 1 \qquad \forall\, t \quad \text{———— ②}$$

Stochastic Hydrology

- One equation in the set ① is redundant in each period $t$ (in the light of set ②); thus the number of independent equations including ② equals the number of variables.

# Stochastic Dynamic Programming

- The unknown probabilities, $PR_{kit}$, are the steady state joint probabilities of the initial storage being in class $k$ and inflow being in class $i$ in period $t$.
- The marginal probabilities of storage and inflow are obtained as

$$PS_{k\,t} = \sum_{i} PR_{ki\,t} \qquad \forall\, k,\, t$$

$$PQ_{i\,t} = \sum_{k} PR_{ki\,t} \qquad \forall\, i,\, t$$

# Example – 1 (Contd.)

Steady state policy for period 1 and period 2

$t = 1$

| $k$ | $i$ | $l^*$ |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 2 | 2 |
| 2 | 1 | 2 |
| 2 | 2 | 2 |

$t = 2$

| $k$ | $i$ | $l^*$ |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 2 | 1 |
| 2 | 1 | 1 |
| 2 | 2 | 1 |

Inflow transition probabilities:

| | $t = 2$ $j$ | | |
|---|---|---|---|
| $t = 1$ | $i$, | 1 | 2 |
| | 1 | 0.5 | 0.5 |
| | 2 | 0.3 | 0.7 |

| | $t = 1$ $j$ | | |
|---|---|---|---|
| $t = 2$ | $i$, | 1 | 2 |
| | 1 | 0.4 | 0.6 |
| | 2 | 0.8 | 0.2 |

# Example – 1 (Contd.)

$$PR_{ljt+1} = \sum_k \sum_i PR_{kit} P_{ij}^t \qquad \sum_k \sum_i PR_{kit} = 1 \quad \forall t$$

Selective Sum

$$l = l^*(k,i,t)$$

$l$ = 2, $j$ = 1, $t$ = 1; Results from three combinations of ($k$,$i$) : (1,2), (2,1) and (2,2)

$t = 1$:

$PR_{112} = PR_{111}$ x 0.5

$PR_{122} = PR_{111}$ x 0.5

$PR_{212} = PR_{121}$ x 0.3 + $PR_{211}$ x 0.5 + $PR_{221}$ x 0.3

$PR_{222} = PR_{121}$ x 0.7 + $PR_{211}$ x 0.5 + $PR_{221}$ x 0.7

$t = 2$:

$PR_{111} = PR_{112}$ x 0.4 + $PR_{122}$ x 0.8 + $PR_{212}$ x 0.4

$PR_{121} = PR_{112}$ x 0.6 + $PR_{122}$ x 0.2 + $PR_{212}$ x 0.6

$PR_{211} = PR_{222}$ x 0.8

$PR_{221} = PR_{222}$ x 0.2

# Example – 1 (Contd.)

$$PR_{111} + PR_{121} + PR_{211} + PR_{221} = 1$$
$$PR_{112} + PR_{122} + PR_{212} + PR_{222} = 1$$

Total 8 probabilities, $PR_{kit}$, to be obtained, 8 equations are required.

Any three equations for $t = 1$, any three equations for $t = 2$ and the last two equations are considered.

| | $k$ | $i$ | $PR_{kit}$ | | $k$ | $i$ | $PR_{kit}$ |
|---|---|---|---|---|---|---|---|
| $t = 1$ | 1 | 1 | 0.284 | $t = 2$ | 1 | 1 | 0.142 |
| | 1 | 2 | 0.284 | | 1 | 2 | 0.142 |
| | 2 | 1 | 0.346 | | 2 | 1 | 0.284 |
| | 2 | 2 | 0.086 | | 2 | 2 | 0.432 |

# Example – 1 (Contd.)

Storage: $PS_{kt} = \sum_i PR_{kit} \qquad \forall\ k,t$

$t = 1$

$PS_{11} = PR_{111} + PR_{121}$

= 0.284+ 0.284

= 0.568

$t = 2$

$PS_{12} = PR_{112} + PR_{122}$

= 0.142+0.142

= 0.284

$PS_{21} = PR_{211} + PR_{221}$

= 0.346+0.086

= 0.432

$PS_{22} = PR_{212} + PR_{222}$

= 0.284 + 0.432

= 0.716

Similarly, for inflows

$PQ_{11} = 0.63; PQ_{21} = 0.37; PQ_{12} = 0.426; PQ_{22} = 0.574$