INDIAN INSTITUTE OF SCIENCE

Water Resources Systems:
Modeling Techniques and Analysis

Lecture - 35

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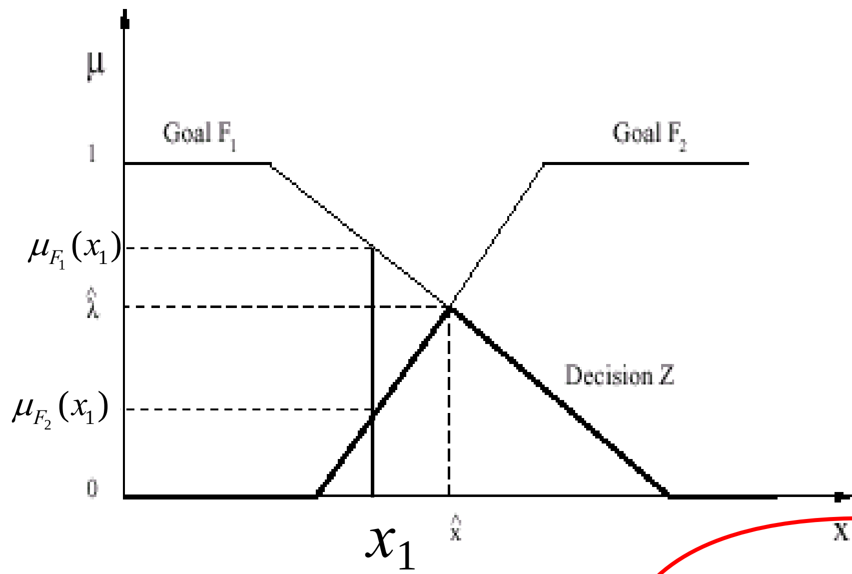
Summary of the previous lecture

- Fuzzy optimization



Max λ

s.t. $1 + b_i'' - \left(B' x\right)_i \geq \lambda \quad \forall\, i$

$x_j \geq 0 \quad \forall\, j$

Example – 1

Crisp problem

Max $Z = 3x_1 + 5x_2$

s.t.

$$x_1 \leq 4$$

$$2x_2 \leq 12$$

$$3x_1 + 2x_2 \leq 18$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

<u>Solution</u> :

$x_1 = 2.0$; $x_2 = 6.0$
$Z = 36$

Example – 1 (Contd.)

Fuzzy LP :

Max λ

s.t.

$$1 + b_i^{''} - \left(B^{'}x\right)_i \geq \lambda \quad \forall\, i$$

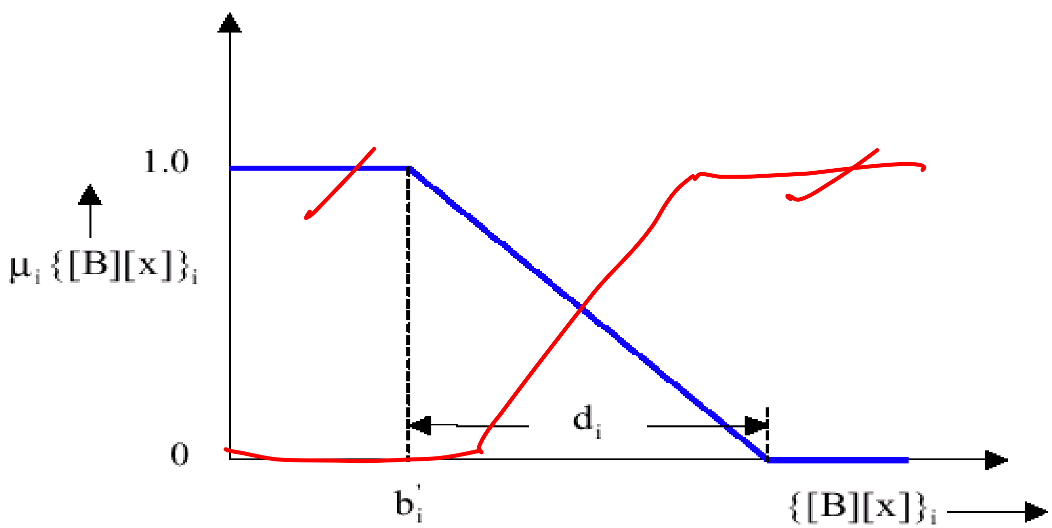
$$x_j \geq 0 \quad \forall\, j$$

⇒

Max λ

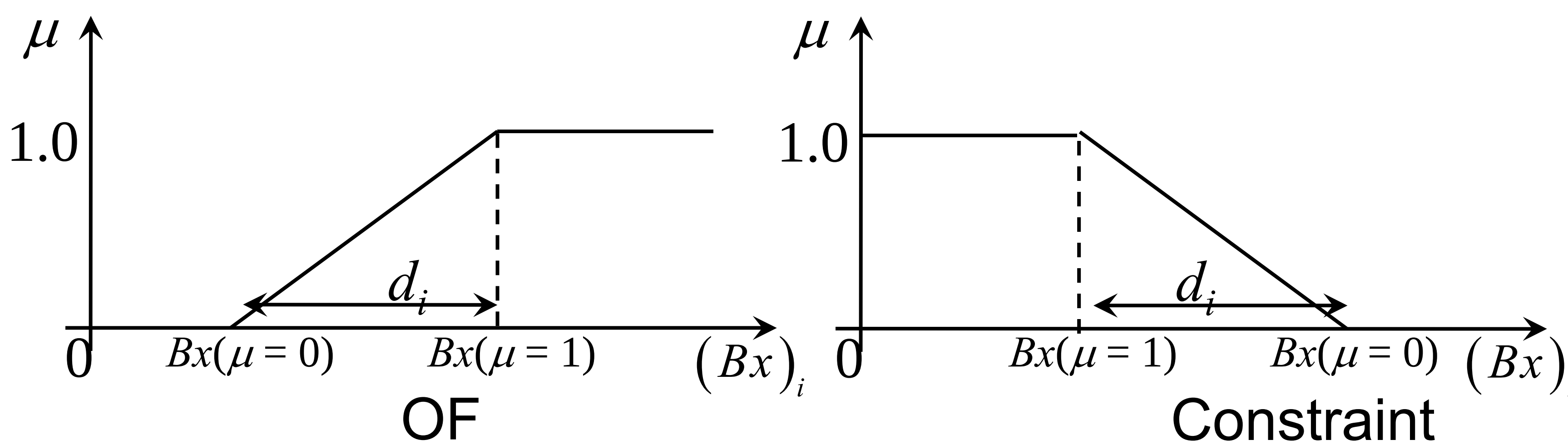
s.t. $\mu\left(Bx\right)_i \geq \lambda \quad \forall\, i$

$$x_j \geq 0 \quad \forall\, j$$



Example – 1 (Contd.)

	$\mu = 0$	$\mu = 1$
O.F	36	38
Cons. 1	6	4
Cons. 2	10	6
Cons. 3	25	18

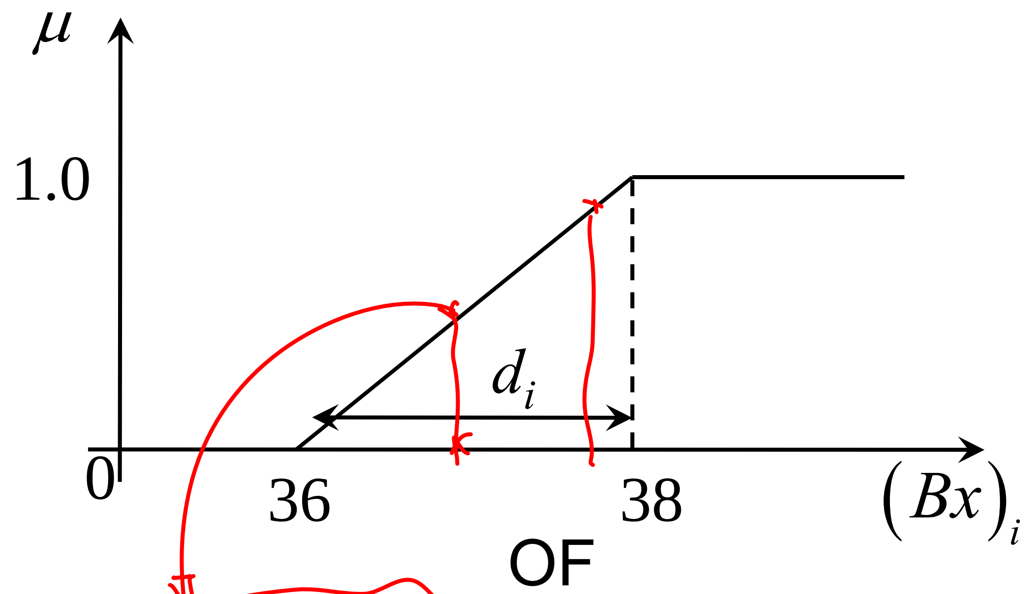


Example – 1 (Contd.)

The first constraint of fuzzy LP (corresponding to O.F. of the original problem) is written as,

$$\mu(Bx)_i \geq \lambda \quad \forall i$$

$$d_i = 38 - 36 = 2$$



$$\mu(Bx)_i = \frac{(Bx)_i - 36}{2} = \frac{(3x_1 + 5x_2) - 36}{2}$$

M.F. for OF.

Example – 1 (Contd.)

$$\mu\left(Bx\right)_i \geq \lambda$$

$$\frac{3x_1 + 5x_2 - 36}{2} \geq \lambda$$

$$1.5x_1 + 2.5x_2 - 18 \geq \lambda$$

Fuzzy constraint for O.F.

Example – 1 (Contd.)

For the first constraint,

$$\mu(Bx)_i \geq \lambda \quad \forall i$$

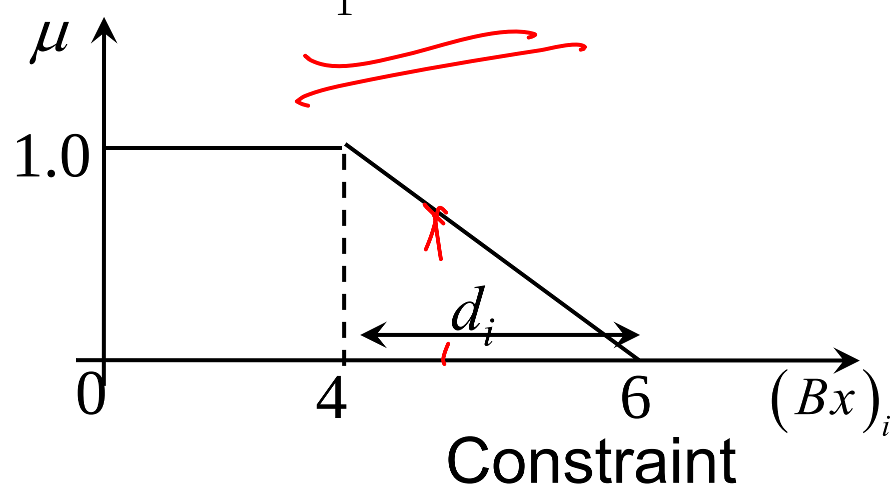
$$d_i = 6 - 4 = 2$$

$$\mu(Bx)_i = \frac{6 - x_1}{2}$$

$$\frac{6 - x_1}{2} \geq \lambda$$

$$3 - 0.5x_1 \geq \lambda$$

$x_1 \leq 4$



Example – 1 (Contd.)

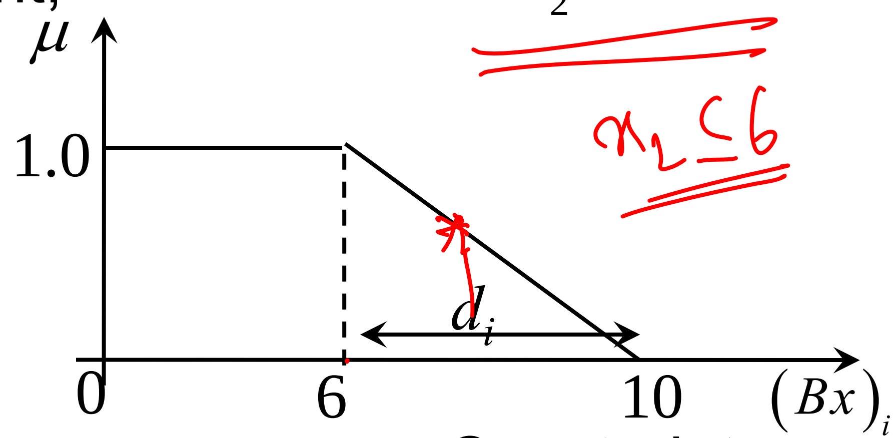
For the second constraint, $2x_2 \le 12$

$x_2 \le 6$

$$\mu(Bx)_i \ge \lambda \quad \forall i$$

$$d_i = 10 - 6 = 4$$

$$\mu(Bx)_i = \frac{10 - x_2}{4}$$



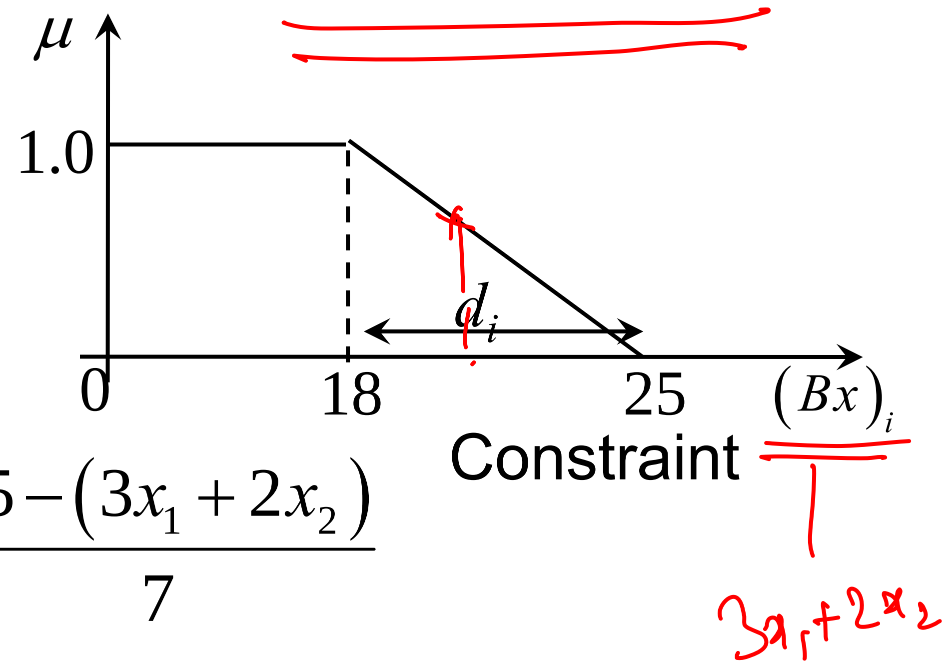
$$\frac{10 - x_2}{4} \ge \lambda$$

$$2.5 - 0.25x_2 \ge \lambda$$

Example – 1 (Contd.)

For the third constraint,

$$3x_1 + 2x_2 \le 18$$



$$\mu(Bx)_i \ge \lambda \quad \forall i$$

$$d_i = 25 - 18 = 7$$

$$\mu(Bx)_i = \frac{25 - (Bx)_i}{7} = \frac{25 - (3x_1 + 2x_2)}{7}$$

$$\frac{25 - (3x_1 + 2x_2)}{7} \ge \lambda$$

$$3.57 - 0.43x_1 - 0.286x_2 \ge \lambda$$

Example – 1 (Contd.)

Crisp equivalent of fuzzy LP

Max λ

s.t. $1.5x_1 + 2.5x_2 - 18 \geq \lambda$

$3 - 0.5x_1 \geq \lambda$

$2.5 - 0.25x_2 \geq \lambda$

$3.57 - 0.43x_1 - 0.286x_2 \geq \lambda$

$x_1 \geq 0; \quad x_2 \geq 0$

Example – 1 (Contd.)

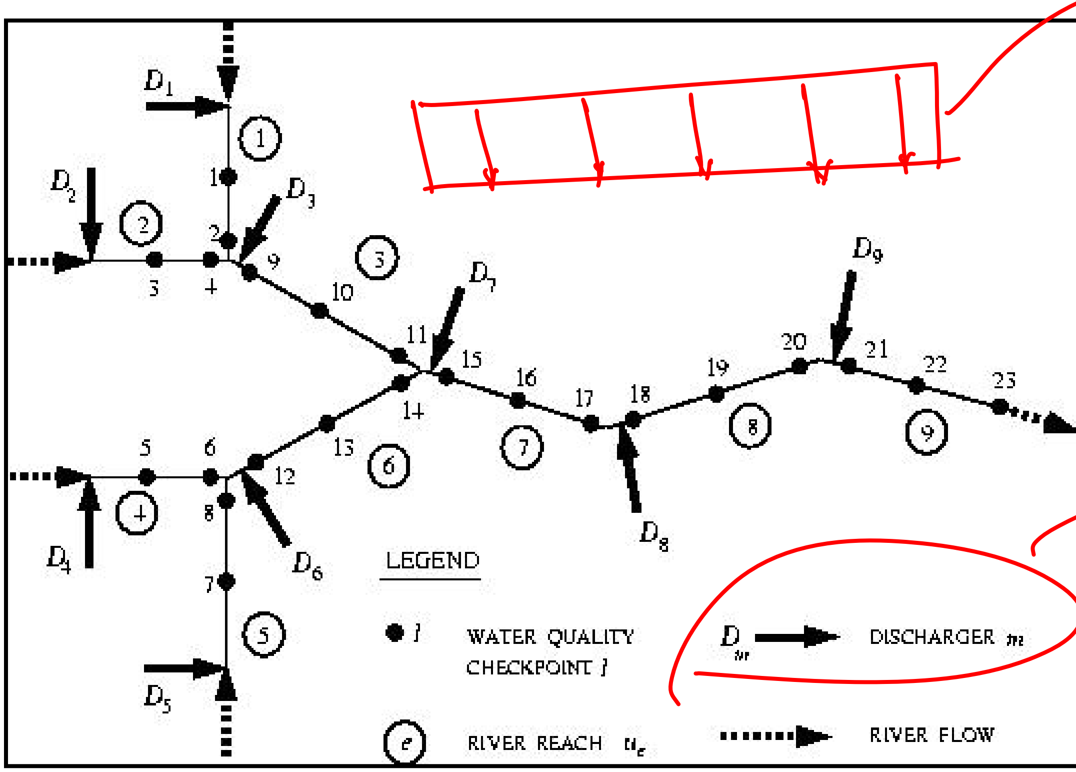
Solution

<u>Non Fuzzy</u>	<u>Fuzzy</u>
$x_1 = 2.0$	$x_1 = 1.95$
$x_2 = 6.0$	$x_2 = 6.39$
$Z = 36$	$Z = 37.8$

- Fuzzy LP allows latitude in constraints
- Instead of maximizing (or minimizing) an objective function, a level of satisfaction for permissible values is defined

Fuzzy Optimization

Stream Water Quality Control Problems



Non-point Source
Point pollution sources
Effluent (Municipal / Industrial)

Specific Objective

To obtain best compromise solutions for effluent fraction removal levels | treatment levels

Fuzzy Optimization

Uncertainties due to randomness and fuzziness

- Randomness in Streamflow, Effluent Flow, Temperature and Reaction Rates
- Fuzziness due to water quality standards, goals & objectives, and nonpoint source pollution

Mathematical Concepts and Tools:

- Fuzzy Decision; Stochastic Optimization; Fuzzy Probabilities; Fuzzy Risk; Fuzzy Inference Systems (FIS)

Fuzzy Optimization

- Concentration level of the water quality parameters i at the checkpoint l is denoted as C_{il}.
- The pollution control agency sets a desirable level, $C^D{}_{il}$, and a minimum permissible level, $C^L{}_{il}$, for the water quality parameter i at the checkpoint l ($C^L{}_{il} > C^D{}_{il}$).

Fuzzy Goals for Water Quality Management:

- The quantity of interest is the concentration level, C_{il}, of the water quality parameter, and the fraction removal level (treatment level), x_{imn}, of the pollutant.
- The quantities x_{imn} are the fraction removal levels of the pollutant n from the discharger m to control the water quality parameter i.

Fuzzy Optimization

Fuzzy Goals of the Pollution Control Agency

- Fuzzy Goal E_{il} : Make the concentration level, C_{il}, of the water quality parameter i at the checkpoint l as close as possible to the desirable level, C^{D}_{il} so that the water quality at the checkpoint l is enhanced with respect to the water quality parameter i, for all i and l.

Fuzzy Goals of the Dischargers

- Fuzzy Goal F_{imn} : Make the fraction removal level x_{imn} as close as possible to the aspiration level x^{L}_{imn} for all i, m, and n.

Fuzzy Optimization

The membership function corresponding to the decision Z is given by

$$\mu_Z(X) = \underset{i,m,n}{minimum}\left[\mu_{E_{il}}(C_{il}),\ \mu_{F_{imn}}(x_{imn})\right]$$

where X is the space of alternatives composed of C_{il} and x_{imn}.

The corresponding optimal decision, X*, is given by

$$\mu_Z(X^*) = \lambda^* = \underset{y}{max}\left[\mu_Z(X)\right]$$

Fuzzy Optimization

Membership Functions for the Fuzzy Goals

Goal E_{il} : The membership function for the fuzzy goal E_{il} is constructed as follows.

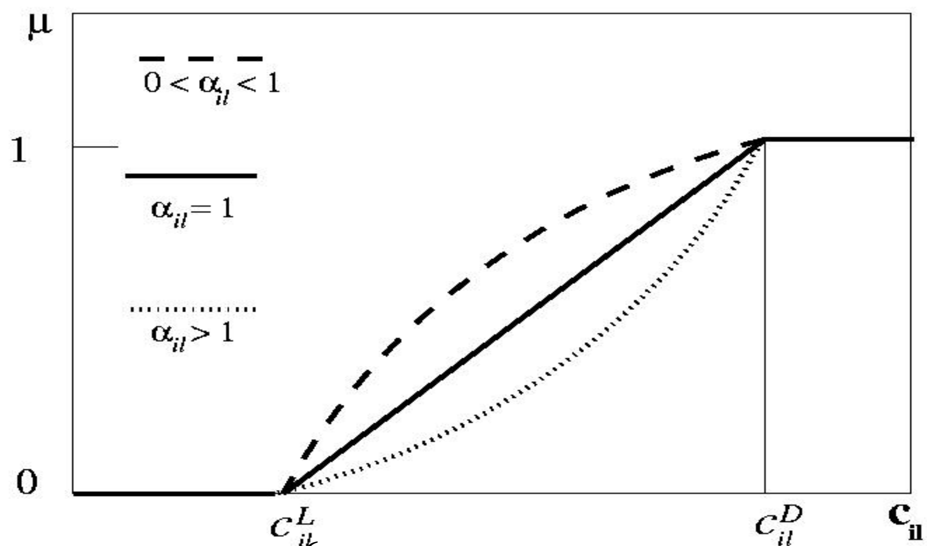
- The desirable level, $C^{D}{}_{il}$, for the water quality parameter i at checkpoint l is assigned a membership value of 1.
- The minimum permissible level, $C^{L}{}_{il}$, is assigned a membership value of zero

$$\mu_{E_{il}}\left(c_{il}\right)=\begin{cases}0 & c_{il}\le c_{il}^{L}\\ \left[\dfrac{c_{il}-c_{il}^{L}}{c_{il}^{D}-c_{il}^{L}}\right]^{\alpha_{il}} & c_{il}^{L}\le c_{il}\le c_{il}^{D}\\ 1 & c_{il}\ge c_{il}^{D}\end{cases}$$

Fuzzy Optimization

FWLAM

Fuzzy Membership Function --- PCA



$$\mu_{E_{il}}(c_{il}) = \begin{cases} 0 & c_{il} \le c_{il}^{L} \\ \left[\dfrac{c_{il} - c_{il}^{L}}{c_{il}^{D} - c_{il}^{L}} \right]^{\alpha_{il}} & c_{il}^{L} \le c_{il} \le c_{il}^{D} \\ 1 & c_{il} \ge c_{il}^{D} \end{cases}$$

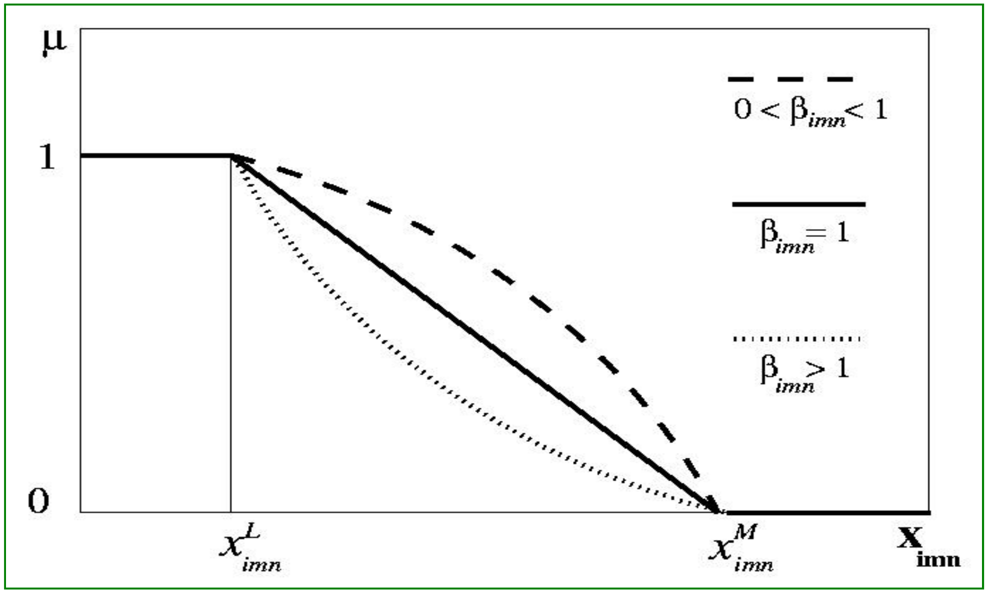
Fuzzy Optimization

With a similar argument, the membership function for the goal F_{imn} is written as

$$\mu_{F_{imn}}\left(x_{imn}\right)=\begin{cases} 0 & x_{imn} \le x_{imn}^{L} \\ \left[\dfrac{x_{imn}^{M}-x_{imn}}{x_{imn}^{M}-x_{imn}^{L}}\right]^{\beta_{imn}} & x_{imn}^{L} \le x_{imn} \le x_{imn}^{M} \\ 1 & x_{imn} \ge x_{imn}^{M} \end{cases}$$

Fuzzy Optimization

Fuzzy Membership Function -- Dischargers



$$\mu_{F_{imn}}(x_{imn}) = \begin{cases} 1 & x_{imn} \le x^{L}_{imn} \\ \left[\dfrac{x^{M}_{imn} - x_{imn}}{x^{M}_{imn} - x^{L}_{imn}}\right]^{\beta_{imn}} & x^{L}_{imn} < x_{imn} < x^{M}_{imn} \\ 0 & x_{imn} \ge x^{M}_{imn} \end{cases}$$

Fuzzy Optimization

Fuzzy multiobjective optimization model (MAX-MIN formulation)

Maximize λ

s.t. $\mu_{Fi}(X) \geq \lambda$ $\forall i$ (fuzzy constraints)

$g_j(X) \leq 0$ $\forall j$ (crisp constraints)

$0 \leq \lambda \leq 1$

λ : Interpreted as the degree of goal fulfillment level

Membership fn.

X: Vector of treatment levels

Fuzzy Optimization

Fuzzy optimization model for FWLAM

Maximize λ

s.t. $\mu_{E_{il}}(c_{il}) \geq \lambda \quad \forall\, i, l$ ....PCA

$\mu_{F_{imn}}(x_{imn}) \geq \lambda \quad \forall\, i, m, n$ ....Dischargers

$c_{il}^{L} \leq c_{il} \leq c_{il}^{D} \quad \forall i, l$

$x_{imn}^{L} \leq x_{imn} \leq x_{imn}^{M} \quad \forall\, i, m, n$

$x_{imn}^{MIN} \leq x_{imn} \leq x_{imn}^{MAX} \quad \forall\, i, m, n$

$0 \leq \lambda \leq 1$

Fuzzy Optimization

- The concentration level, C_{il}, of the water quality parameter i at the checkpoint l can be related to the fraction removal level, x_{imn}, of the pollutant *n* from the discharger m to control the water quality parameter i, though the transfer function that may be mathematically expressed as

$$C_{il} = \sum_{m=1}^{N_d} \sum_{n=1}^{N_p} f_{ilmn}(L_{ilmn},\ x_{imn}) + \sum_{p=1}^{N_t} \sum_{n=1}^{N_p} f_{ilpn}(L_{ilpn})$$

where L_{ilmn} is the concentration of the pollutant n prior to treatment from the discharger m that affects the water quality parameter i at the checkpoint l,

L_{ilpn} is the concentration of the pollutant n from the uncontrollable source p that affect the water quality parameter i at the checkpoint l.

Fuzzy Optimization

- The transfer functions $f_{ilmn}(\cdot, \cdot)$ and $f_{ilpn}(\cdot)$ represent the concentration levels of the water quality parameter i due to $L_{ilmn}(1 - x_{imn})$, and L_{ilpn} respectively
- These transfer functions can be evaluated using appropriate mathematical models that determine the spatial and temporal distribution of the water quality parameter due to the pollutants in the river system
- The solution of the optimization model are X^* and λ^* where X^* is the set of optimal fraction removal levels, and λ^* is the maximized m

Example – 2

Consider a hypothetical river network as shown