CSL 852, Computational Geometry: Practice Problems

Clustering point sets using quadtrees and applications

1. Let $S = \{p_1, p_2 \dots p_n\}$ be a set of n points in the plane and let k be a positive integer.We would like to cover all the points using k disks of diameter D, such that D is minimum - let it be denoted by D_o.

   (a) Let $q_1, q_2 \dots q_{k+1}$ be a subset of S such that for all $j \geq 1$, $d_j \geq d_{j+1}$ where $d_j = \max_i ||q_{j+1} - q_i|| \quad i = \{1, 2 \dots j\}$.
   Argue that the $D_o \geq ||q_{k+1} - q_k||$.

   Solution: Consider the alternate definition that the points in $q_1 \dots q_{k+1}$ are separated at least by distance d. Then any optimal solution will consist of k disks such that some disk must contain at least 2 points (from pigeon hole argument). Therefore $D_o \geq d$.

   (b) Let the points q_i be defined in the following way. Start from an arbitrary point $q_1 \in S$. Let q_2 be the furthest point from q_1 and for $1 < i \leq k+1$, $q_i = \max_{p \in S} d(p, Q_{i-1})$ where $Q_{i-1} = \{q_1, q_2 \dots q_{i-1}\}$ and $d(p, Q)$ denotes the distance from p to its closest neighbour in Q.
   Prove that you can cover the points of S using k disks of radius D_o, thereby obtaining a factor 2 approximation algorithm.

   Solution: Clearly all points are within a distance of d_k from Q_{k-1} (as d_k is the furthest distance neighbour of Q_{k-1}). So the disks of radius d_k (diameter $2d_k$) centered at Q_{k-1} cover all points.

   To prove the approximation bound, first you must **prove** that the points q_i defined this way satisfy the conditions of the part (a). Let $d_{i-1} = d(q_i, Q_{i-1}) \quad i > 1$. By induction show that $d_{j+1} \leq d_j$. So, $D_o \geq d_k$ and the covering disks have diameter $2d_k$.

2. The weight of an εWSPD is defined as $\sum_i^k (|A_i| + |B_i|)$ where $\{(A_1, B_1) \dots (A_k, B_k)\}$ are the pairs of the WSPD. For a fixed ε, construct a set of points P such that the weight of a valid WSPD of P has weight $\Omega(n^2)$.

   Solution: Given ε, consider n points on a line at coordinates $\alpha, 2\alpha \dots 2^{n-1}\alpha$ where $\alpha = \lceil 1/\varepsilon \rceil$. We have to show that for this configuration of points, the WSPD will always have weight $\Omega(n^2)$ whatever be the algorithm used. Let p_1, p_2 be the points in increasing order.

   For this point set, all the WSP (A_i, B_i) must be such that they are contained in non-overlapping intervals (otherwise distance is 0). Moreover, if the points in B are larger than A then $|B| = 1$. Otherwise the diameter is larger than the separation. The weight of such a pair is $|A_i|$ and it covers exactly $|A_i|$ pairs, therefore the total weight must be at least $\Omega(n^2)$ for covering all pairs.

   Note that it is not sufficient to show that some WSPD has weight $\Omega(n^2)$ as that is always true for the trivial WSPD.