Introduction to Formal Languages, Automata and Computability

Finite State Automata : Output and Minimization

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Myhill-Nerode Theorem

In an earlier chapter we considered some examples of FSA. Consider the example serial adder. After getting some input, the machine can be in 'carry' state or 'no carry' state. It does not matter what exactly the earlier input was. It is only necessary to know whether it has produced a carry or not. Hence the finite state automaton need not distinguish between each and every input. It distinguishes between classes of inputs.

contd.

In the above case, the whole set of inputs can be partitioned into two classes - one that produces a carry and another that does not produce a carry. Thus the finite state automaton distinguishes between classes of input strings. These classes are also finite . Hence we say that the FSA has finite amount of memory.

Myhill-Nerode

Theorem The following three statements are equivalent.

1. $L \subseteq \Sigma^*$ is accepted by a DFSA.
2. L is the union of some of the equivalence classes of a right invariant equivalence relation of finite index on Σ^*.
3. Let equivalence relation R_L be defined over Σ^* as follows: xR_Ly if and only if, $\forall z \in \Sigma^*$, xz is in L exactly when yz is in L. Then R_L is of finite index.

Proof

We shall prove (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3), (3) $\Rightarrow$ (1).
(1) $\Rightarrow$ (2).
Let L be accepted by a FSA $M = (K, \Sigma, \delta, q_0, F)$.
Define a relation R_M on Σ^* such that xR_My if $\delta(q_0, x) = \delta(q_0, y)$. R_M is an equivalence relation, as seen below.
$\forall\ x\ xR_Mx$, since $\delta(q_0, x) = \delta(q_0, x)$,
$\forall\ x\ xR_My \Rightarrow yR_Mx \because \delta(q_0, x) = \delta(q_0, y)$ means $\delta(q_0, y) = \delta(q_0, x)$,
$\forall\ x, y\ xR_My$ and $yR_Mz \Rightarrow xR_Mz$. For if $\delta(q_0, x) = \delta(q_0, y)$ and $\delta(q_0, y) = \delta(q_0, z)$ then $\delta(q_0, x) = \delta(q_0, z)$.

contd.

So R_M divides Σ^* into equivalence classes. The set of strings which take the machine from q_0 to a particular state q_i are in one equivalence class. The number of equivalence classes is therefore equivalent to the number of states of M, assuming every state is reachable from q_0. (If a state is not reachable from q_0, it can be removed without affecting the language accepted). It can be easily seen that this equivalence relation R_M is right invariant, i.e., if
$xR_My, xzR_Myz \; \forall z \in \Sigma^*$.
$\delta(q_0, x) = \delta(q_0, y)$ if xR_My,
$\delta(q_0, xz) = \delta(\delta(q_0, x), z) = \delta(\delta(q_0, y), z) = \delta(q_0, yz)$.
Therefore xzR_Myz.

contd.

L is the union of those equivalence classes of R_M which correspond to final states of M.

(2) $\Rightarrow$ (3)

Assume statement (2) of the theorem and let E be the equivalence relation considered. Let R_L be defined as in the statement of the theorem. We see that $xEy \Rightarrow xR_Ly$.

If xEy, then $xzEyz$ for each $z \in \Sigma^*$. xz and yz are in the same equivalence class of E. Hence xz and yz are both in L or both not in L as L is the union of some of the equivalence classes of E. Hence xR_Ly.

Hence any equivalence class of E is completely contained in an equivalence class of R_L. Therefore E is a refinement of R_L and so the index of R_L is less than or equal to the index of E and hence finite.

contd.

(3) $\Rightarrow$ (1)
First we show R_L is right invariant. xR_Ly if $\forall z$ in Σ^*, xz is in L exactly when yz is in L or we can also write this in the following way: xR_Ly if for all w, z in Σ^*, xwz is in L exactly when ywz is in L.
If this holds xwR_Lyw.
Therefore R_L is right invariant. Let $[x]$ denote the equivalence class of R_L to which x belongs.
Construct a DFSA $M_L = (K', \Sigma, \delta', q_0, F')$ as follows: K' contains one state corresponding to each equivalence class of R_L. $[\epsilon]$ corresponds to q_0'. δ' is defined as follows: $\delta'([x], a) = [xa]$. This definition is consistent as R_L

contd.

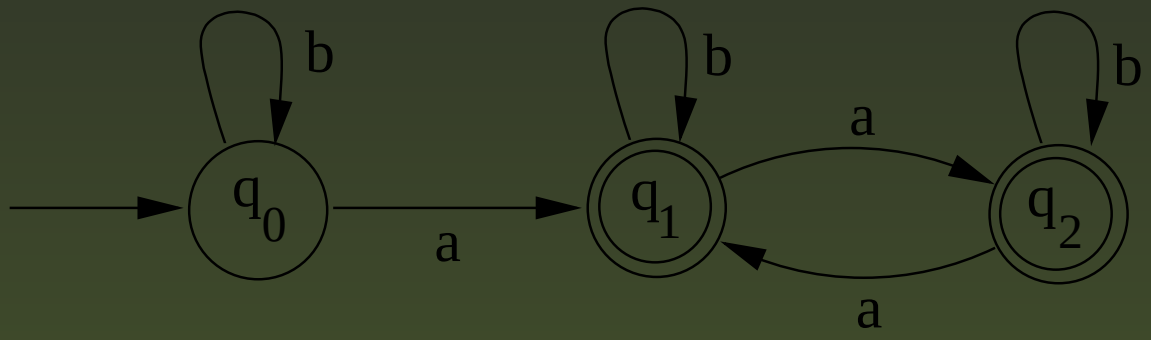
is right invariant. Suppose x and y belong to the same equivalence class of R_L. Then xa and ya will belong to the same equivalence class of R_L. For,

$$\begin{array}{ccc} \delta'([x], a) & = & \delta'([y], a) \\ \Downarrow & & \Downarrow \\ [xa] & = & [ya] \end{array}$$

if $x \in L$, $[x]$ is a final state in M', i.e., $[x] \in F'$. This automaton M' accepts L.

Example

Consider the FSA M given in next figure.



The language accepted consists of strings of a's and b's having at least one a. M divides $\{a, b\}^*$ into 3 equivalence classes.
1. H_1, set of strings which take M from q_0 to q_0 i.e., b^*.
2. H_2, set of strings which take M from q_0 to q_1, i.e., set of strings which have odd numbers of a's.

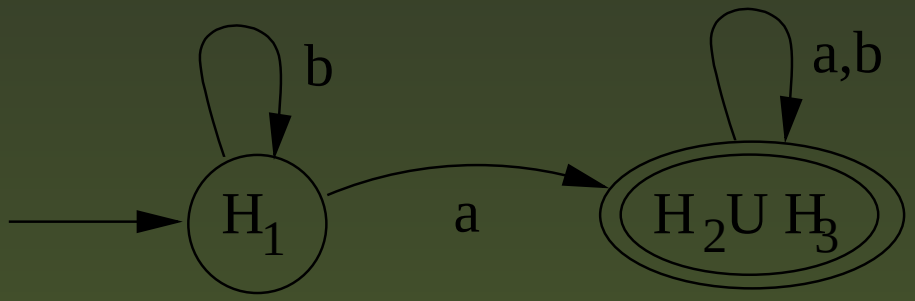
contd.

3. H_3, set of strings which take M from q_0 to q_2, i.e., set of strings which have even number of a's.
$L = H_2 \cup H_3$ as can be seen.

1. Let $x \in H_1$ and $y \in H_2$. Then $xb \in H_1$ and $yb \in H_2$. Then $xb \notin L$ and $yb \in L$. Therefore $x \not R_L y$.
2. Let $x \in H_1$ and $y \in H_3$. Then $xb \in H_1$ and so $xb \notin L$ and $yb \in H_3$ and so $xb \in L$. Therefore $x \not R_L y$.
3. Let $x \in H_2$ and $y \in H_3$. Take any string z, xz belongs to either H_2 or H_3 and so in L, yz belongs to either H_2 or H_3 and so in L. Therefore $x R_L y$.

contd.

So if we construct M' as in the proof of the theorem, we have one state corresponding to H_1 and one state corresponding to $L = H_2 \cup H_3$.



This is the automaton we get as M'. We see that, it accepts $L(M)$. Both M and M' are DFSA accepting the same language. But M' has minimum number of states and is called the minimum state automaton.

Theorem The minimum state automaton accepting a regular set L is unique up to an isomorphism and is given by M' in the proof of previous theorem.

In the proof of previous theorem, we started with M, found equivalence classes for R_M, R_L and constructed M'. The number of states of M is equal to the index of R_M and the number of states of M' is equal to the index of R_L. Since R_M is a refinement of R_L, the number of states of M' is less than or equal to the number of states of M. If M and M' have the same number of states, then we can find a mapping $h : K \rightarrow K'$

contd.

(which identifies each state of K with a state of K') such that if $h(q) = q'$ then for $a \in \Sigma$,

$$h(\delta(q, a)) = \delta'(q', a).$$

This is achieved by defining h as follows: $h(q_0) = q_0'$ and if $q \in K$, then there exists a string x such that $\delta(q_0, x) = q$. $h(q) = q'$ where $\delta(q_0', x) = q'$. This definition of h is consistent. This can be seen as follows: Let $\delta(q, a) = p$ and $\delta'(q', a) = p'$, $\delta(q_0, xa) = p$ and $\delta'(q_0', xa) = p'$ and hence $h(p) = p'$.

Minimization of DFSA

Let $M = (K, \Sigma, \delta, q_0, F)$ be a DFSA. Let R be an equivalence relation on K such that pRq, if and only if for each input string x, $\delta(p, x) \in F$ if and only if $\delta(q, x) \in F$. This essentially means that if p and q are equivalent, then either $\delta(p, x)$ and $\delta(q, x)$ both are in F or both are not in F for any string x. p is distinguishable from q if there exists a string x such that one of $\delta(q, x)$, $\delta(p, x)$ is in F and the other is not. x is called the distinguishing string for the pair $< p, q >$.

If p and q are equivalent $\delta(p, a)$ and $\delta(q, a)$ will be equivalent for any a. If $\delta(p, a) = r$ and $\delta(q, a) = s$ and r and s are distinguishable by x, then p and q are distinguishable by ax.

Algorithm to find minimum DFSA

We get a partition of the set of states of K as follows:
Step 1 Consider the set of states in K. Divide them into two blocks F and $K - F$. (Any state in F is distinguishable from a state in $K - F$ by ϵ)
Repeat the following step till no more split is possible.
Step 2 Consider the set of states in a block. Consider the a-successors of them for $a \in \Sigma$. If they belong to different blocks, split this block into two or more blocks depending on the a-successors of the states.

For example if a block has $\{q_1, \ldots, q_k\}$. $\delta(q_1, a) = p_1$, $\delta(q_2, a) = p_2, \ldots, \delta(q_k, a) = p_k$ and $p_1, \ldots, p_i$ belong to one block, $p_{i+1}, \ldots, p_j$ belong to another block and

contd.

$p_{j+1}, \ldots, p_k$ belong to third block, then split $\{q_1, \ldots, q_k\}$ into $\{q_1, \ldots, q_i\}$ $\{q_{i+1}, \ldots, q_j\}$ $\{q_{j+1}, \ldots, q_k\}$.

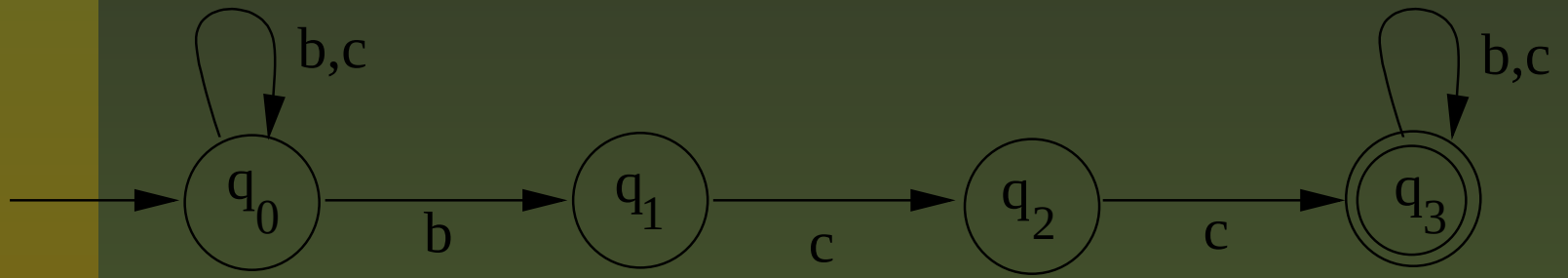
Step 3 For each block B_i, consider a state b_i. Construct $M' = (K', \Sigma, \delta', q_0', F')$ where $K' = \{b_i | B_i$ is a block of the partition obtained in step 2$\}$.

q_0' corresponds to the block containing q_0.

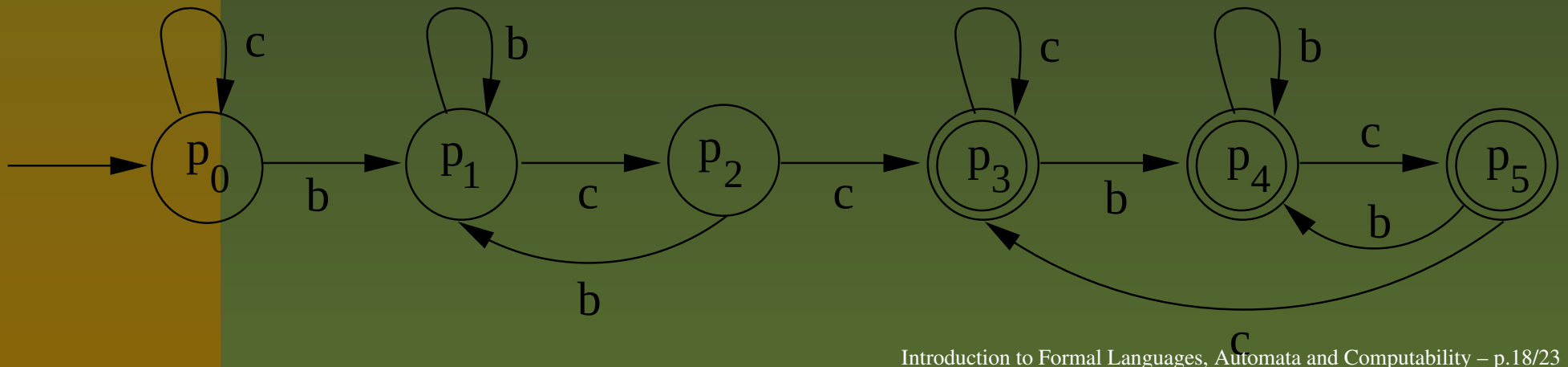
$\delta(b_i, a) = b_j$ if there exits $q_i \in B_i$ and $q_j \in B_j$ such that $\delta(q_i, a) = q_j$. F' consists of states corresponding to the blocks containing states in F.

Example

Consider the following FSA M over $\Sigma = \{b, c\}$ accepting strings which have bcc as a substrings. A nondeterministic automaton for this will be,



Converting to DFSA we get M' as in next figure.



contd.

where, $p_0 = [q_0] \quad p_1 = [q_0, q_1] \quad p_2 = [q_0, q_2]$
$p_3 = [q_0, q_3] \quad p_4 = [q_0, q_1, q_3]$
$p_5 = [q_0, q_2, q_3]$. Finding the minimum state automaton for M'

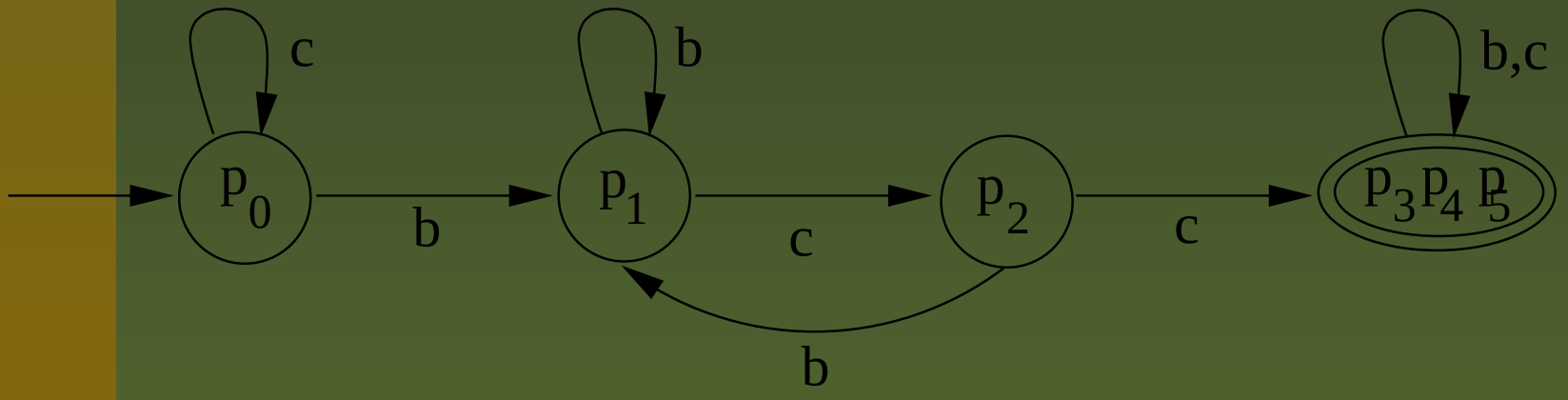
Step 1 Divide the set of states into 2 blocks

$$\overline{p_0 p_1 p_2} \quad \overline{p_3 p_4 p_5}.$$

In $\overline{p_0 p_1 p_2}$, the b successors are in one block, the c successors of $p_0 p_1$ are in one block and p_2 is in another block. Therefore $\overline{p_0 p_1 p_2}$ is split as $\overline{p_0 p_1}$ and $\overline{p_2}$. The b and c successors of $\overline{p_3 p_4 p_5}$ are in the same block.

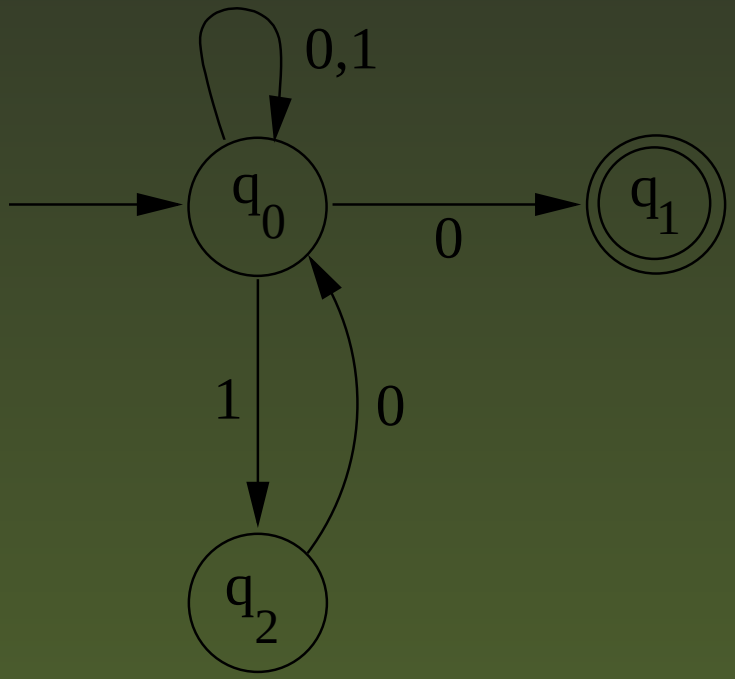
contd.

Now the partition is $\overline{p_0 p_1}$ $\overline{p_2}$ $\overline{p_3 p_4 p_5}$. Consider $\overline{p_0 p_1}$. The b successors of $\overline{p_0 p_1}$ are in the same block but the c successors of $\overline{p_0}$ and $\overline{p_1}$ are $\overline{p_0}$ and $\overline{p_2}$ and they are in different blocks. Therefore $\overline{p_0 p_1}$ is split into $\overline{p_0}$ and $\overline{p_1}$. Now the partition is $\overline{p_0}$ $\overline{p_1}$ $\overline{p_2}$ $\overline{p_3 p_4 p_5}$. No further split is possible. The minimum state automaton is



contd.

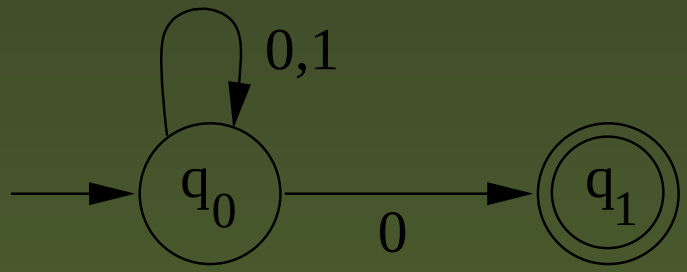
The minimization procedure cannot be applied to NFSA. For example consider the NFSA



The language accepted is represented by the regular expression $(0 + 1)^*0$ for which NFSA in next figure will

contd.

suffice. But if we try to use the minimization procedure $\overline{q_0q_1q_2}$ will be initially split as $\overline{q_0q_2}$ and $\overline{q_1}$. q_0 and q_2 are not equivalent as $\delta(q_0, 0)$ contains a final state while $\delta(q_2, 0)$ does not. So they have to be split and the FSA in first figure cannot be minimized using the minimization procedure.



Myhill-Nerode theorem can also be used to show that certain sets are not regular.

Example

We know $L = \{a^n b^n | n \geq 1\}$ is not regular. Suppose L is regular. Then by Myhill-Nerode theorem, L is the union of the some of the equivalence classes of a right invariant relation $\approx$ over $\{a, b\}$. Consider $a, a^2, a^3, \ldots$ since the number of equivalence classes is finite, for some m and n, $m \neq n$, a^m and a^n must be in the same equivalence class. We write this as $a^m \approx a^n$. Since $\approx$ is right invariant $a^m b^m \approx a^n b^m$. i.e., $a^m b^m$ and $a^n b^m$ are in the same equivalence class. L either contains one equivalence class completely or does not contain that class. Hence since $a^m b^m \in L$, L should contain this class completely and hence $a^n b^m \in L$ which is a contradiction. Therefore L is not regular.