Module 1: Covering problems

1. Let G be a graph such that the number cycles in G of length at most g is at most $n/2$. Let the cardinality of the minimum vertex cover, $\beta(G) = k$. Then show that there exists a subgraph G' of G with $\chi(G') \geq \frac{n}{2(n-k)}$ and girth $> g$.

2. Let A be an $n \times n$ matrix where the entry at the (i, j)th position a_{ij} is either 1 or 0 for $1 \leq i, j \leq n$. Let A_{ij} denote the $(n-1) \times (n-1)$ matrix obtained from A by deleting the ith row and jth column. Define the permanant of A, per(A) as follows:

$$\text{per}(A) = \sum_{j=1}^{n} a_{1j}.\text{per}(A_{1j})$$

   Let G_A denote the bipartite graph with one part being $\{u_1, \ldots, u_n\}$ and the other part being $\{v_1, \ldots, v_n\}$, and u_i is adjacent to v_j if and only if $a_{ij} = 1$ in the matrix A.

   Show that per(A) = the total number of perfect matchings in G_A.

3. Consider the matrix A defined in the previous question. Show that perm$(A) = 0$ if and only if A contains an $s \times t$ zero submatrix such that $s + t = n + 1$.

4. Consider a complete r-partite graph G having even number of vertices and with parts $A_1, A_2, \ldots, A_r$. Assume that $|A_1| \geq |A_2| \geq \cdots \geq |A_r| \geq 1$. Show that there exists a perfect matching in G if and only if $|A_1| \leq \sum_{i=2}^{r} |A_i|$, using Tutte's theorem.

5. Show that in a bipartite graph a minimum vertex cover is a "barrier". (A set $S \subseteq V$ is a barrier if there exists a matching M such that the number of unmatched vertices with respect to M equals $q(G-S) - |S|$, where $q(G-S)$ is the number of odd components of $G - S$.

6. Let G be a bipartite graph with parts X and Y where $|X| = |Y|$. Let $d = \max(|S| - |N(S)| : S \subseteq X)$. Show that $\alpha'(G) = |X| - d$.

7. Let $G = (A \cup B, E)$ be a bipartite graph. For a matching M of G, let $S_M^A = \{u \in A : u \text{ is a matched vertex with respect to } M\}$. We call a matching M a special matching iff there exists no other matchings M' such that $S_M^A \subset S_{M'}^A$. Show that if M is a special matching then M is a maximum matching of G. (Your proof should be from the basics.)

8. Let G be a directed acyclic graph (DAG). Then show that any mimimal (directed) path cover of G also has to be a minimum (directed) path cover.

   (Hint: (Use if necessary) Construct a bipartite graph -the cover graph- corresponding to the directed graph as we discussed in class.)

9. Let G be k-regular bipartite graph. Then show that every edge (u, v) of G belongs to some perfect matching of G.

10. Given a connected undirected graph $G = (V, E)$ (where $|V| \geq 2$) a dominating set S is a subset of V such that for every $u \in V - S$, there exists a $v \in S$ such that u is adjacent to v. Let $D(G)$ denote the cardinality of the minimum dominating set in G. Then show that

    (a) $D(G) \leq MVC(G)$, where MVC(G) is the cardinality of the minimum vertex cover. Are there disconnected graphs where this is not true ? Explain.

    (b) Is there a fixed constant $c \leq 1$ such that $c.MVC(G) \leq D(G)$, for every connected graph G ? Prove your answer.

11. Let $G = (V, E)$ be an undirected graph. Show that G is bipartite if and only if, there exists a subset S of V such that both S and $V - S$ are vertex covers of G.

12. Let G be an undirected graph obtained from a complete graph on n nodes (let n be a sufficiently large integer) by removing 45 edges. Let G' be a directed graph obtained by orienting the edges of G. Then show that G' contains a directed path on at least $\frac{n}{10}$ nodes.

13. Let G be a connected undirected graph such that every edge of G belongs to some perfect matching of G. Then show that G can not have a cut vertex.

14. Let G be a bipartite graph, and let $\Delta(G) \geq 1$ be its maximum degree. Show that G contains a set of independent edges such that each vertex of degree $\Delta(G)$ is incident with at least one edge in this set.

15. Show that if G has a perfect matching (i.e., 1–factor), the number of vertices of G (i.e., order of G) is at least $2k+2$, and every set of k independent edges is contained in some 1–factor, then every set of $k - 1$ independent edges is contained in some 1–factor.

16. Let G be a graph of order at least 4 and let $F = \{f_1, f_2, \cdots, f_m\}$ be a 1–factor of G. Show that F contains two edges (a_i, b_i) and (a_j, b_j) say, such that $G - \{a_i, b_i\}$ and $G - \{a_j, b_j\}$ are both connected.

17. Let G be a bipartite graph with n vertices. Consider a maximum matching M of G. Let C be the set of unmatched vertices of G with respect to M. Let $\mathcal{E}$ be the set of vertices of G which are reachable from some vertex in C by an alternating path of even length, where length of the path is the number of edges in it. Similarly let $\mathcal{O}$ be the set of vertices of G reachable from C by an alternating path of odd length.

    (a) Show that $\mathcal{E} \cap \mathcal{O} = \emptyset$.

    (b) What is the cardinality of M, in terms of $|\mathcal{O}|$ and $|\mathcal{E}|$ and n ? Prove your answer.

Module 2: Connectivity

1. What is the vertex connectivity κ of $K_{m,n}$, the complete bipartite graph with m and n vertices on the two parts. Explain your answer.

2. Let G be a simple graph of diameter two. Show that the edge connectivity of G is equal to its minimum degree, i.e. $\lambda = \delta(G)$.

3. (a) Show that if G is simple and the minimum degree $\delta(G) \geq n - 2$, (n being the number of vertices in G) then the vertex connectivity $\kappa(G) = \delta(G)$.

   (b) For each $n \geq 4$, find a simple graph with $\delta(G) = n - 3$ and $\kappa(G) < \delta(G)$.

4. Show that if G is simple, with $n \geq k + 1$, and $\delta(G) \geq (n + k + 2)/2$, then G is k-connected.

5. Show that if d is sufficiently large, then d-dimensional hypercube H_d is $\lfloor d/2 \rfloor$-linked, using the following Theorem of Kühn and Osthus: "For every $s \in N$, there exists a $k_s \in N$ such that if $K_{s,s}$ is not a subgraph of G and $\kappa(G) \geq 2k \geq k_s$ then G is k-linked." (Here $\kappa(G)$ is the vertex connectivity of G, and $K_{s,s}$ is the complete bipartite graph with s vertices on each side.) Also estimate the best possible lower bound on d in terms of k_s (choosing the best possible value for s: assume that k_s increases with s) so that your proof (based on Kühn-Osthus theorem) works.

6. Show from basic principles (at least do not use Kühn and Osthus theorem) a reasonable lower bound for k in terms of d, so that H_d is k-linked. (For example you may want to try to show that $k \geq \lfloor d/3 \rfloor$. A possible hint– only if you need it– is to imitate the proof done in the class for the general case. You may earn up to 10 marks extra for this question if you get a better bound than most others get or if you have an interesting method)

7. Show that a connected graph G is a complete graph if and only if G does not contain any induced subgraph isomorphic to $2K_2$ (i.e. just two disjoint edges) or a P_3, (a path on 3 vertices).

8. Show that the vertex connectivity of d-dimensional hypercube is d.

9. Let G be an undirected k–regular graph for an odd integer k, and let its edge connectivity be at least $k - 1$. Then show that G has a perfect matching.

Module 3: Coloring

1. Let $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_m\}$. Consider a graph G defined as follows: $V(G)$ is a subset of $X \times Y$ and two distinct vertices $[x_i, y_j], [x_a, y_b] \in V(G)$ are adjacent if and only if: either $x_i = x_a$ or $y_j = y_b$. For $x_i \in X$, let $\gamma(x_i) = |(\{x_i\} \times Y) \cap V(G)|$ and for $y_j \in Y$, let $\gamma(y_j) = |(\{y_j\} \times X) \cap V(G)|$. Now get an expression for the chromatic number $\chi(G)$ of G in terms of the function γ. Prove your answer.

2. Let $K_{n,n,n}$ denoted the complete tri-partite graph with n vertices in each part.

   (a) When $n \geq 1$ is an odd integer, what is the edge chromatic number of $K_{n,n,n}$, i.e. $\chi'(K_{n,n,n})$? Prove your answer.

   (b) Find a proper edge coloring of $K_{2,2,2}$ using 4 colors.

   (c) Show that $\chi'(K_{n,n,n}) = 2n$, when $n \geq 2$ is even. (Hint. You may try to use part (b) of this question.)

3. Consider a drawing G' of a (not necessarily planar) graph G in the plane. Two edges of G' cross if they meet at a point other than a vertex of G'. Each such point is called a crossing of the two edges. The crossing number of G, denoted by $cr(G)$, is the least number of crossings in a drawing of G in the plane. Show that, $cr(K_5) = 1$ and $cr(K_{3,3}) = 1$.

4. Show that the crossing number satisfies the inequality $cr(G) \geq m - 3n + 6$ provided that $n \geq 3$.

5. Let G be a connected graph with girth k, where $k \geq 3$. Show that the number of edges, $m \leq k(n-2)/(k-2)$.

6. Consider the vertex coloring problem: we need to give a color to each vertex in the graph making sure that no two adjacent vertices get the same color. Given a graph G, the chromatic number of G is defined to be the minimum positive integer k such that we can color the vertices of G with k colors as described above. Show that if G is a co-comparability graph (i.e. the compliment of a comparability graph) then the biggest complete subgraph (clique) in G has exactly k vertices.

7. Consider the d-dimensional hypercube H_d. Recall that the vertices of H_d corresponds to the 2^d, d-dimensional 0-1 vectors, and two vertices are adjacent if and only if the hamming distance of the corresponding vectors is exactly 1. Show that H_d is non-planar for each $d \geq 4$.

8. What is the chromatic number of H_d, the d-dimensional hypercube.

9. Show that for a simple graph G,

$$\text{arboricity}(G) \leq \text{degeneracy}(G) \leq 2\ \text{arboricity}(G) - 1$$

.

10. Consider a graph $G = (V, E)$ and for each $v \in V$, let $C(v)$ denote the set of maximal cliques of G containing the vertex v. Consider a coloring of the vertices of a graph satisfying the following constraint: u and v can get the same color, only if either $C(u) \subseteq C(v)$ or $C(v) \subseteq C(u)$. Let $f(G)$ denote the minimum number of colors required to color the vertices of G in this way. Show that $f(G) \geq \chi(\overline{G})$.

11. Consider a coloring c of the vertices of G with the following properties (1) c is a proper vertex coloring of G; (2) u and v can get the same color only if either $N(v) \subseteq N(u)$ or $N(u) \subseteq N(v)$. Let $g(G)$ be the minimum number of colors required so that G can be colored in this way. What is the relation between the two parameters $f(.)$ and $g(.)$ (where $f(.)$ is as defined in the previous problem).

Module 4: Special classes of graphs

1. Let G be an interval graph: That is to each vertex $v \in V(G)$, we can associate an interval $I(v)$ on the real line such that two vertices u and v are adjacent if and only if $I(v) \cap I(u) \neq \emptyset$. Show that $\chi(G) = \omega(G)$, where $\omega(G)$ is the clique number of G.

2. Let G be a non-trivial simple graph with degree sequence $(d_1, d_2, \ldots, d_n)$ where $d_1 \leq d_2 \leq \ldots \leq d_n$. Suppose that there is no integer $k < (n+1)/2$ such that $d_k < k$ and $d_{n-k+1} < n-k$. Show that G has a hamiltonian path.

3. A graph G is called self-complementary if it is isomorphic to $\overline{G}$, its complement. Give an example of a self-complementary graph.

4. Prove that every self-complementary graph has a hamiltonian path.

5. Let $P = \{p_0, p_1, \ldots, p_{n-1}\}$ be a set of n distinct points on the plane. Let $r_0, \ldots, r_{n-1}$ be positive real numbers. Let (p_i, r_i) represent the cycle centered at p_i and of radius r_i. Let us define a simple graph $G = (V, E)$ with $|V| = 2n$ as follows: Let $V = \{v_0, v_1, \cdots, v_{2n-1}\}$ and $f : V \rightarrow P$ be such that $f(v_i) = p_{i \bmod n}$. In G let v_i and v_j (for $i \neq j$) be adjacent if and only if the two cycles (p_i, r_i) and (p_j, r_j) intersect. Let G' be the induced subgraph of G on the vertex set $V' = \{v_0, v_1, \ldots, v_{n-1}\} \subset V$. Show that G is a perfect graph if and only if G' is a perfect graph. (Give a complete argument).

6. A graph G is a self-complementary graph iff G is isomorphic to its complement $\overline{G}$. Show that any regular self-complementary graph has a hamiltonian path.

7. Let G be a simple graph of minimum degree δ. Show that, G contains a path of length 2δ if G is connected and $\delta \leq (n-1)/2$.

8. Let G be any simple finite non-planar graph. Let H be a graph obtained from G by replacing each edge of G by a path of legth 3. (Length of a path equals the number of edges in the path.) Then, show that

   (a) $\text{box}(H) \geq 3$.

   (b) $\text{box}(H) \leq 3$.

9. Show that the chordal dimension of a complete k partite graph (each part having at least 2 vertices) is k.

10. Show that chordal dimension of a graph G is at most $\chi(G)$.

11. Show that $\text{cub}(P_n \times P_n \times \cdots \times P_n)$ is $\Omega(\frac{d \log n}{\log d})$. Here P_n is the path on n vertices, and the product is taken d times.

12. Show that boxicity of Peterson graph is at most 3.

13. Construct two graph G_1 and G_2 such that, G_1 is not isomorphic to G_2, but $L(G_1)$ is isomorphic to $L(G_2)$, where $L(G)$ denotes the line graph of G.

Module 5: Network flows

1. In a large university with k academic departments, we must appoint an important committee. One professor will be chosen from each department. Some professors have joint appointments in two or more departments, but each must be the designated representative of at most one department. We must use equally many assistant professors, associate professors and full professors among the chosen representatives (assume that k is divisible by 3.) Build a network in which units

of flow corresponds to professors chosen for the committee and capacities enforce the various constraints. Explain how to use your network to test whether such a committee exists and find it if it exists.

2. Let D be a digraph, and let f be a real-valued function on A. Show that f is a circulation in D if and only if, $f^+(X) = f^-(X)$ for all $X \subseteq V$.

3. Let $P = (p_1, p_2, \ldots, p_m)$ and $Q = (q_1, q_2, \ldots, q_n)$ be two sequences of non-negative integers. The pair (P, Q) is said to be realizable by a simple bipartite graph if there exists a simple bipartite graph with bipartition $\{x_1, x_2, \ldots, x_m\}$ and $\{y_1, y_2, \ldots, y_n\}$ such that degree$(x_i) = p_i$ for $1 \leq i \leq m$ and degree$(y_i) = q_i$ for $1 \leq i \leq n$. Formulate as a network flow problem, the problem of determining whether a given pair (P, Q) is realizable by a simple bipartite graph. Explain your answer.

4. Show that: (a) If a graph G has a k-flow, then some orientation of G has a positive k-flow. (b) A connected digraph has a positive k-flow for some $k \geq 1$, if and only if it is strongly connected.

5. Show that any graph which admits a 2-flow is even.

Module 6: Random graphs and probabilistic method

1. Let $X \geq 0$ be a random variable on $\mathcal{G}(n,p)$ and $a > 0$. Then prove that $Pr(X \geq a) \leq \frac{E(X)}{a}$.

2. Show that for a sufficiently large positive integer n and every real number $p \in (0, 1]$, there exists a graph on n vertices with at most $(np)^3$ triangles and stability number at most $\left\lceil \frac{2 \log n}{p} \right\rceil$.

3. Prove that the Ramsey number $R(3, k)$ (i.e the minimum integer t such that for any graph G on at least t vertices, there is either a K_3 or an independent set of cardinality k) satisfies the inequality $R(3, k) > n - (np)^3$, for any sufficiently large integer n and every real number $p \in (0, 1]$ where $k = \left\lceil \frac{2 \log n}{p} \right\rceil + 1$ (Hint: Make use of the previous question.)

4. Let $G = (V, E)$ be a d-regular graph on n vertices. Show that we can partition the vertices of G into $O(d/\log d)$ subsets say $X_1, X_2, \ldots, X_k$, such that for each vertex $v \in V$, $|N(v) \cap X_i| \leq c \log d$, for some constant c. (Use Lovasz Local Lemma).

5. Let K_n denote the complete graph on n nodes. Let $\theta(n)$ be the cardinality of a MINIMUM bipartite graph cover of G. (That is, a minimum collection of bipartite subgraphs of K_n, such that every edge of K_n is in at least one of these bipartite graphs- i.e. covered by one of these bipartite graphs.) Show that $\theta(n) \leq 2 \log n$, using a simple probabilistic argument.

6. Let G be a simple finite undirected graph with average degree $d > 1$. Consider the following experiment: Let S be an empty set initially. Toss a coin for each vertex of G, and add the vertex to set S if (and only if) you get HEAD. Let E_S denote the set of edges of G with both end points in S. Clearly $|S|$ and $|E_S|$ are random variables. Answer the following questions.

(a) Find the expectation of $|S|$ and $|E_S|$ in terms of n and p and d, where n is the number of vertices in G and p is the probability of getting HEAD when we toss the coin.

(b) Show that in G, there is an independent set of cardinality at least $\frac{n}{2d}$. (Use the information from the above problem- try to give an appropriate value for p).

7. Let π be a random permutation of $1, 2, \cdots, n$. (That is a permutation is picked uniformly at random from the $n!$ possible permutations.) Now consider the following algorithm.

   $T = 0$

   For $i = 1$ to n,

   If $\pi(i) > T$ then $T = \pi(i)$.

   Let the random variable X denote the number of assignments that take place in line 3 of the algorithm, when it is executed with the random permutation π as input. What is the expectation of X ?

8. Let π be a permutation of $1, 2, \ldots, n$, chosen uniformly at random from the $n!$ possible permutations. i is said to be a fixed point of π if $\pi(i) = i$. What is the expected number of fixed points in π ? Prove your answer.

9. Let X be a random variable with $E(X) = \mu$ and $\mathrm{Var}(X) = \sigma^2$. Let $X_1, X_2, \ldots, X_k$ be random variables, that are independently and identically distributed as X. Let $Y = \frac{\sum_{i=1}^{k} X_i}{k}$. Then show that

$$Pr(|Y - \mu| > t) \leq \frac{\sigma^2}{t^2 m}$$

Module 7: Graph minors

1. Let G be simple, finite graph such that

$$\sum_{v \in V(G)} degree(v) < 90$$

.

   Then prove that K_{10}, the complete graph on 10 vertices cannot be a minor of G.

2. Recall that we proved in class the following theorem. "Suppose that H is a graph with maximum degree $\Delta(H) \leq 3$. Then for any graph G, H is a minor of G if and only if H is a topological minor of G." Now prove that there exists no graph H with maximum degree $\Delta(H) \geq 4$ such that the same statement is true, i.e. "For any graph G, H is a minor of G if and only if H is a topological minor of G."

3. The biggest positive integer n such that K_n is a topological minor of G is the Hajos number of G. Show that for a d-dimensional hyper cube, the Hajos number is $d+1$.

4. Let $G = P_n \times P_n \times K_2$, where P_n is a path on n vertices and $\times$ stands for the cartesian product operation. Show that K_{n+1} is a minor of G.

5. Show that for any graph G, $\chi(G) \leq treewidth(G) + 1$.