# Graph Theory: Lecture No. 1

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# References


1. Reinhard Diestel : Graph Theory (Springer)
2. Douglas B. West: Introduction to Graph Theory (Prentice-Hall India)
3. A. Bondy and U.S.R. Murty: Graph Theory (Springer)
4. B. Bollabas : Modern Graph Theory (Springer)

# What is a Graph ?

It is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$ and a relation that associates with each edge two vertices (not necessarily distinct) called its end points.

- A loop
- Multiple edge.
- Simple Graph
- Finite Graph

# Some simple graphs

- Complete Graph.
- Cycle
- Path

# Subgraph and Induced Subgraph

- $H$ is a subgraph of $G$: Then $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$. The assignment of end points to edges in $H$ is the same as that in $G$.
- $H$ is an induced subgraph of $G$ on $S$, where $S \subseteq V(G)$: Then $V(H) = S$ and $E(H)$ is the set of edges of $G$ such that both the end points belong to $S$.

A graph $G$ is connected if each pair of vertices belongs to a path.

# Isomorphism

An Isomorphism from a simple graph $G$ to a simple graph $H$ is a bijection $f : V(G) \to V(H)$ such that $(u, v) \in E(G)$ if and only if $(f(u), f(v)) \in E(H)$

# Forest and Tree

- A graph without any cycle is acyclic.
- A forest is an acyclic graph
- A tree is a connected acyclic graph.

# Bipartite Graph

A graph $G$ is bipartite if $V(G)$ is the union of two disjoint (possibly empty) independent sets called partite sets of $G$.
(A subset $S \subseteq V(G)$ is an independent set if the induced subgraph on $S$ contains no edges. )

- A tree is a bipartite graph.
- Can we say that the complete graph $K_n$ is bipartite ?
- The complete bipartite graph.

# Vertex Cover

A set $S \subseteq V(G)$ is a vertex cover of $G$ (of the edges of $G$) if every edge of $G$ is incident with a vertex in $S$.
A vertex cover of the minimum cardinality is called a minimum vertex cover. We will denote this set by MVC(G).

- What is the cardinality of MVC in the complete graph $K_n$ ?
- What about the complete bipartite graph $K_{m,n}$ ?
- The cycle $C_n$, when $n$ is even and odd ?

The cardinality of a biggest independent set in $G$ is called the independence number (or stability number) of $G$ and is denoted by $\alpha(G)$.

Is there any relation between $|MVC(G)|$ and $\alpha(G)$ ?

- If we remove a VC from $G$, the rest is an independent set.
- So, if we remove MVC from $G$, the rest, i.e. $V - MVC$ is an independent set.
- So, $\alpha(G) \geq n - |MVC(G)|$. Thus $|MVC(G)| \geq n - \alpha(G)$.
- Similiarly if we remove any independent set from $G$, the rest is VC, and so $|MVC| \leq n - \alpha(G)$.
- Thus we get $|MVC| = n - \alpha(G)$.
- If we denote —MVC(G)— by $\beta(G)$, then we have $\beta(G) + \alpha(G) = n$.

# Matching

- A set $M$ of independent edges in a graph is called a matching.
- $M$ is a matching of $U \subseteq V(G)$, if every vertex in $U$ is incident with an edge in $M$.
- Then a vertex in $U$ is a matched vertex. The vertices which are not incident with any edge of $M$ is unmatched.

A matching $M$ is a perfect matching of $G$, if every vertex in $G$ is matched by $M$.
If $G$ has $n$ vertices, what is the cardinality of a perfect matching $M$ of $G$ ?

The cardinality of the biggest matching in $G$ can be denoted by $\alpha'(G)$.

What is the value of $\alpha'(G)$ for:

- Cycle $C_n$
- Path $P_n$
- Complete Graph $K_n$
- Complete Bipartite graph $K_{m,n}$