Graph Theory: Lecture No. 2

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- **A set M of independent edges in G is called a matching.**
- **Matched vertex**
- **Unmatched vertex**

The cardinality of the biggest matching in G can be denoted by $\alpha'(G)$.

What is the value of $\alpha'(G)$ for:

- **Cycle** C_n
- **Path** P_n
- **Complete Graph** K_n
- **Complete Bipartite graph** $K_{m,n}$

If every vertex of G is matched with respect to a matching M, then it is called a perfect matching.

How many edges are there in a perfect matching, if G has n vertices ? What can we tell about n ?

- **A perfect matching is also known as a 1-factor.**
- **A k-factor is a k-regular spanning subgraph of G.**
- **What can we tell about a 2-factor.**

- **In general do we have any relation between $\alpha'(G)$ and $\alpha(G)$?**
- $\alpha(G) \geq n - 2\alpha'(G)$
- **So, do we have any relation between the minimum vertex cover and maximum matching ?**
- $n - \beta(G) \geq n - 2\alpha'(G)$
- $\alpha'(G) \leq \beta(G) \leq 2\alpha'(G)$

A stronger relation holds in bipartite graphs (König, 1931)

> **For a bipartite graph G, the maximum cardinality of a matching is equal to the minimum cardinality of its vertex cover**

Suppose König's statement, namely $\beta(G) = \alpha'(G)$ is true, for bipartite graphs. And we are trying to come up with a proof.

Suppose M is a matching such that $|M| = \alpha'(G)$.
For proving the theorem we will try to demonstrate a vertex cover S, with $|S| = \alpha'(G)$

S should be such that it contains exactly one point from each edge of M

So we see that we are forced to add some edges in S. Let us try to understand this.

An alternating path: A path that starts at an unmatched vertex in A and then contains alternately edges from $E - M$ and M.
If an alternating path ends at an unmatched vertex, then it is called and augmenting path.
An augmenting path starts from unmatched vertex on the A side, and ends at an unmatched vertex on the B side.
If we can find in G and Augmenting path with respect M, then M is not a maximum matching.

Hall's Condition:
For all $S \subseteq A$**,** $|N(S)| \geq |S|$**.**

Hall's Theorem

A bipartite graph G has a matching of A if and only if G satisfies Hall's condition

Using Hall's Theorem:

If G is k-regular ($k \geq 1$) bipartite graph, then it has a perfect matching