Graph Theory: Lecture No. 3

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Hall's Theorem: Second Proof.

- **Suppose Hall's condition is satisfied, and there is matching of A.**
- **Then there is an unmatched vertex in A. Call it a_0.**
- **We will generate a sequence of distinct vertices,** $a_0, b_1, a_1, b_2, a_2, \ldots$

- **In this sequence b_i, a_i will always be a matching edge.**
- b_{i+1} **will be a neighbor of some vertex in** $\{a_0, a_1, \ldots, a_i\}$
- **Start with a_0. We can always find b_1. Can b_1 be an unmatched vertex ?**

- **After picking a_0 to a_i, can we get b_{i+1}, always ? Yes- because of Hall's condition.**
- **Can b_{i+1} be an unmatched vertex ? No- in that case we will get an augmenting path.**
- **Since b_{i+1} is a matched vertex, we can always get a_{i+1}.**
- **So, the sequence never ends. But since there are only finite number of vertices, it is a contradiction.**

Using Hall's Theorem:

If G is k-regular ($k \geq 1$) bipartite graph, then it has a perfect matching

To show this we just need to show that the Hall's condition is true for k-regular bipartite graphs.

Any $2k$-regular graph has a cycle cover (2-factor).

Before getting to the proof the above statement, we need to discuss some concept. First, given an undirected graph here is a way of associating a bipartite graph to it.
Bipartite Double Cover of G: The two sides A and B are copies of $V(G)$. Lets us say the copy of the vertex i of G, is named a_i in A and b_i in B.
We add an edge from a_i to b_j whenever there is an edge (i, j) in G.

In the above, note that if (i, j) is an edge in G, we get two different edges (a_i, b_j) and (a_j, b_i). For a directed graph, if we do a corresponding construction, a directed edge will correspond to exactly one edge in the bipartite cover graph.

Relation between a directed cycle cover in a directed graph and a perfect matching in its bipartite double cover.

To show that an undirected $2k$-regular graph G has a cycle cover (2-factor) we covert G to a bipartite graph as follows.
First we give a direction to each edge of G to get a directed graph G'. Now get the bipartite cover $B_{G'}$ of G'.

- **Instead of showing a cycle cover in G, we will show a directed cycle cover in G'.**
- **To show a directed cycle cover in G' it is enough to show a perfect matching in $B_{G'}$.**
- **For that it is enough to show that $B_{G'}$ is regular.**

- **But, is $B_{G'}$ regular ? For $B_{G'}$ to be regular G' should be such that for each vertex v of G, indegree$(v) =$ outdegree$(v) = k$.**
- **So, how do we orient the edges of G, so that G' has the above property ?**
- **The trick is to use and Euler tour.**

- **What is an Euler Tour ?**
- **How does help us to orient the edges of G ?**
- **Finally, is it guaranteed that there is an Euler tour in G?**

A connected graph G has an Euler tour if and only if every vertex has even degree.