Graph Theory: Lecture No. 4

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Tutte's Condition:
(Let G be a graph, and let $q(G)$ be the number of odd components in G.)
For all $S \subseteq V(G)$, $q(G - S) \leq |S|$.

G has a perfect matching (1-factor) if and only if Tutte's condition is satisfied.

It is easy to show that if there is a perfect matching, Tutte's condition is true.
To prove the converse, i.e. if there is no perfect matching, then Tutte's condition is violated (i.e. there exists a set $S \subseteq V(G)$ such that $q(G - S) > |S|$) is more difficult.

A set S, such that $q(G - S) > |S|$ is a bad set.
If G does not have a perfect matching, we have to demonstrate a bad set in G.

Instead of demonstrating a bad set in G, it is enough show a bad set in $G + e$, where e is a new edge. The same bad set set will be bad for G also.
Thus we can choose to demonstrate a bad set in the edge maximal graph G' produced from G by adding new edges, with respect to the property of not having a perfect matching.

Consider a set S with the following special structure. The induced subgraph on S is a clique. Each components C_i of $G - S$ also induces a clique. Between S and C_i for each i, all possible edges are present.
Such a set S is always a bad set in the edge maximal graph G'. So, we only have to look for such a set S.

We take
$S = \{v \in V(G) | v$ **is a universal vertex in** $G\}$**.**
Our intention is to show that S **has the special structure mentioned earlier.**

Every bridge-less cubic graph has a perfect matching.

For any S, each odd component has at least 3 edges going to S. Thus there are at least $q(G-S).3$ edges reaching S. Since the degree of each vertex is only 3, we get $|S|.3 \geq q(G-S).3$ and therefore Tutte's condition is satisfied.