Graph Theory: Lecture No. 5

L. Sunil Chandran

Computer Science and Automation,
Indian Institute of Science, Bangalore
Email: sunil@csa.iisc.ernet.in

G is a factor critical graph if for each $v \in V(G)$, $G - v$ (the graph obtained by removing the vertex v from G) has a perfect matching.

For any graph G, there exists $S \subseteq V(G)$ such that the following two properties are satisfied:

1. **Consider the bipartite graph obtained by contracting each component of $G - S$ and deleting the edges with both end points in S. In this graph, there is a matching of S.**
2. **The induced subgraph on each component is factor critical.**

Given such a subset S, G has a perfect matching if and only if the number of components of $G - S$ equals $|S|$.

The second part of the statement follows immediately from the first part.

If such a set exists in G, then that is a bad set, if G does not have a perfect matching. So, Tutte's theorem follows immediately from the above statement.

Every bridge-less cubic graph has a perfect matching.

For any S, each odd component has at least 3 edges going to S. Thus there are at least $q(G-S).3$ edges reaching S. Since the degree of each vertex is only 3, we get $|S|.3 \geq q(G-S).3$ and therefore Tutte's condition is satisfied.