Graph Theory: Lecture No. 20

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Let G be a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. Set $X = (x_1, \ldots, x_n)$. The adjacency polynomial of G is the multivariate polynomial

$$A(G, X) = \Pi_{i<j}\{(x_i - x_j) : v_i v_j \in E\}$$

Let D **be an orientation of** G**. Then** $\sigma(D) = \Pi\{\sigma(a) : a \in A(D)\}$ **where** $\sigma(a) = +1$ **if** $a = (v_i, v_j)$ **with** $i < j$ **and** $\sigma(a) = -1$ **if** $a = (v_i, v_j)$ **with** $i > j$**.**

Let $d = (d_1, d_2, \ldots, d_n)$ **be a sequence of non-negative integers whose sum is** m**. We define the weight of** d **by**

$$w(d) = \Sigma\sigma(d)$$

where the sum is taken over all orientations D **of** G **whose out degree sequence is** d**.**

Setting $x^d = \Pi_{i=1}^{n} {x_i}^{d_i}$

$$A(G, X) = \Sigma_d w(d) x^d$$

Let f be a nonzero polynomial over a field F in the variables $X = (x_1, x_2, \ldots, x_n)$, of degree d_i in x_i, for $1 \leq i \leq n$. Let L_i be a set of $d_i + 1$ elements of F, $1 \leq i \leq n$. Then there exists $t \in L_1 \times \ldots \times L_n$ such that $f(t) \neq 0$.

THE COMBINATORIAL NULLSTELLENSATZ: Let f be a polynomial over a field F in the variables $x = (x_1, x_2, \ldots, x_n)$. Suppose that the total degree of f is $\sum_{i=1}^{n} d_i$ and that the coefficients in f of $\prod {x_i}^{d_i}$ non-zero. Let L_i be a set of $d_i + 1$ elements of F, $1 \leq i \leq n$. Then there exists a $t \in L_1 \times \ldots \times L_n$ such that $f(t) \neq 0$.

Suppose G has an orientation D such that its outdegree sequence is d, then:

1. **If D' is an orientation of G with outdegree sequence d then $\sigma(D') = \sigma(D)$ if and only if $|A(D) - A(D')|$ is even.**
2. **If D has no directed odd cycles, then all orientations of G with outdegree sequence d have the same sign.**

Let G be a graph and let D be an orientation of G without directed odd cycles. Then G is $(d+1)$-list colorable, where d is the outdegree sequence of G.

$$C(G, k) = C(G - e, k) - C(G/e, k)$$

For any loopless graph G, there exists a polynomial $P(G, x)$ such that $P(G, k) = C(G, k)$ for all non-negative integers k. (Here $C(G, k)$ is the number of distinct proper k-colorings of a graph G) Moreover, if G is simple, and e is any edge of G, then $P(G, x)$ satisfies the recursion formula:

$$P(G, x) = P(G - e, x) - P(G/e, x)$$

The polynomial $P(G, x)$ is of degree n, with integer coefficients which alternate in sign, leading term x^n and constant term 0.