Graph Theory: Lecture No. 21

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Let $d = (d_1, d_2, \dots, d_n)$ be a sequence of non-negative integers whose sum is m. We define the weight of d by

$$w(d) = \Sigma\sigma(D)$$

where the sum is taken over all orientations D of G whose out degree sequence is d.

Setting $x^d = \Pi_{i=1}^{n} {x_i}^{d_i}$

$$A(G, X) = \Sigma_d w(d) x^d$$

Let f be a nonzero polynomial over a field F in the variables $X = (x_1, x_2, \ldots, x_n)$, of degree d_i in x_i, for $1 \leq i \leq n$. Let L_i be a set of $d_i + 1$ elements of F, $1 \leq i \leq n$. Then there exists $t \in L_1 \times \ldots \times L_n$ such that $f(t) \neq 0$.

THE COMBINATORIAL NULLSTELLENSATZ: Let f be a polynomial over a field F in the variables $x = (x_1, x_2, \ldots, x_n)$. Suppose that the total degree of f is $\sum_{i=1}^{n} d_i$ and that the coefficients in f of $\Pi {x_i}^{d_i}$ non-zero. Let L_i be a set of $d_i + 1$ elements of F, $1 \leq i \leq n$. Then there exists a $t \in L_1 \times \ldots \times L_n$ such that $f(t) \neq 0$.

If G has an odd number of orientations D with outdegree sequence d. Then G is $\mathbf{d}+1$ list colourable.

Suppose G has an orientation D such that its outdegree sequence is d, then:

1. **If D' is an orientation of G with outdegree sequence d then $\sigma(D') = \sigma(D)$ if and only if $|A(D) - A(D')|$ is even.**
2. **If D has no directed odd cycles, then all orientations of G with outdegree sequence d have the same sign.**

Let G be a graph and let D be an orientation of G without directed odd cycles. Then G is $(d+1)$-list colorable, where d is the outdegree sequence of G.

$$C(G, k) = C(G - e, k) - C(G/e, k)$$

For any loopless graph G, there exists a polynomial $P(G,x)$ such that $P(G,k) = C(G,k)$ for all non-negative integers k. (Here $C(G,k)$ is the number of distinct proper k-colorings of a graph G) Moreover, if G is simple, and e is any edge of G, then $P(G,x)$ satisfies the recursion formula:

$$P(G,x) = P(G-e,x) - P(G/e,x)$$

The polynomial $P(G,x)$ is of degree n, with integer coefficients which alternate in sign, leading term x^n and constant term 0.

Colouring of Digraphs: Gallai-Roy Theorem: Every digraph D contains a directed path with χ vertices.

Critical Graphs: A graph G is colour critical if $\chi(H) < \chi(G)$ for every proper subgraph H of G. A k-critical graph is one that is k-chromatic and critical.

If G is k-critical, then $\delta \geq k-1$.

No critical graph has a clique cut.

Every critical graph is non-separable.