Graph Theory: Lecture No. 32

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A circulation is digraph D is a function $f : A \to R$, which satisfies the conservation condition at every vertex:

$$f^{+}(v) = f^{-}(v), \forall v \in V$$

Incidence Matrix $M = (m_{va})$ **of a digraph** D**:**
$m_{va} = 1$ **if** v **is the tail of arc** a
$m_{va} = -1$ **if** v **is the head of arc** a
= 0 else

$Mf = 0$ **if** f **is a circulation of** D**, where** M **is the incidence matrix of** D**.**

Let f be a non-zero circulation in a digraph. Then the support of f contains a cycle. Moreover if f is non-negative, then the support of f contains a directed cycle.

Let C be a cycle, together with a given sense of traversal. An arc of C is a forward arc if its direction agrees with the sense of traversal of C and a reverse arc otherwise. We denote the set of forward and reverse arcs by C^+ and C^- respectively and associate with C the circulation f_C defined by $f_c(a) = 1$ if $a \in C^+$, $f_C(a) = -1$ if $a \in C^-$ and $f_C(a) = 0$ otherwise. f_C is indeed a circulation.

Every circulation on a digraph is a linear combination of the circulations associated with its cycle.

Every non-negative circulation in a digraph is a non-negative linear combination of the circulations associated with its directed cycles. Moreover if the circulation is integer values, then the coefficients of the linear combination may be chosen to be non-negative integers.

Let $N = N(x, y)$ **be a network in which each arc is of unit capacity. Then** N **has an** (x, y)**-flow of value** k **if and only if its underlying digraph** $D(x, y)$ **has** k **arc-disjoint directed** (x, y)**-paths.**

Menger's Theorem for Directed graphs: In any digraph $D(x, y)$ the maximum number of pairwise arc disjoint directed (x, y) paths is equal to the minimum number of arcs in an (x, y)-cut.