Graph Theory: Lecture No. 33

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Let C be a cycle, together with a given sense of traversal. An arc of C is a forward arc if its direction agrees with the sense of traversal of C and a reverse arc otherwise. We denote the set of forward and reverse arcs by C^+ and C^- respectively and associate with C the circulation f_C defined by $f_c(a) = 1$ if $a \in C^+$, $f_C(a) = -1$ if $a \in C^-$ and $f_C(a) = 0$ otherwise. f_C is indeed a circulation.

Every circulation on a digraph is a linear combination of the circulations associated with its cycle.

Every non-negative circulation in a digraph is a non-negative linear combination of the circulations associated with its directed cycles. Moreover if the circulation is integer values, then the coefficients of the linear combination may be chosen to be non-negative integers.

Let $N = N(x, y)$ be a network in which each arc is of unit capacity. Then N has an (x, y)-flow of value k if and only if its underlying digraph $D(x, y)$ has k arc-disjoint directed (x, y)-paths.

Menger's Theorem for Directed graphs: In any digraph $D(x, y)$ the maximum number of pairwise arc disjoint directed (x, y) paths is equal to the minimum number of arcs in an (x, y)-cut.

The circulation space $\mathcal{C}$ of a digraph D is the orthogonal complement of the row space of its incidence matrix M.

Let g **be an element of the row space of** M**, i.e.** $g = pM$ **for some vector** $p \in R^V$**. If** $a = (x, y)$ **is an arc, then** $g(a) = p(x) - p(y)$**.**

With each bond B one may associate a tension g_B defined by: $g_B(a) = 1$ if a is a forward arc, and $g_B(a) = -1$ if a is a reverse arc and $g_B(a) = 0$ if $a \notin B$.

Let g be a non-zero tension in a digraph D. Then the support of g contains a bond. Moreover if g is non-negative, then the support of g contains a directed bond.

Every tension in a digraph is a linear combination of the tensions associated with its bonds. Every non-negative tension in a digraph is a non-negative linear combination of the tensions associated with its directed bonds. Moreover, if the tension is integer valued, the coefficients of the linear combination may be chosen to be non-negative integers.

Let $D = (V, A)$ be a directed graph. Suppose that with each arc a of D, two real numbers $b(a)$ and $c(a)$ are associated such that $b(a) \leq c(a)$. A circulation f in D is feasible, with respect to the functions b and c, if $b(a) \leq f(a) \leq c(a)$ for all $a \in A$.

If there is a feasible circulation in D, we should have $c^+(X) \geq b^-(X)$, for all $X \subseteq V$.

Hoffman's Circulation Theorem: A Digraph D has a feasible circulation with respect to bounds b and c if and only if these bounds satisfy the above inequality. Furthermore if b and c are integer valued and satisfy this inequality, then D has an integer valued feasible circulation