Graph Theory: Lecture No. 37

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Let $k > 0$ **be an integer, and let** $p = p(n)$ **be a function of** n **such that** $p \geq (6k \ln n)/n$ **for large** n**. Then** $lim_{n \to \infty} P(\alpha \geq \frac{n}{2k}) = 0$

For every integer k, there exists a graph H with girth $g(H) > k$ and chromatic number $\chi(H) > k$.

- $k \geq 3$, **and fix** $0 < \epsilon < \frac{1}{k}$. **Let** $p = n^{\epsilon - 1}$
- **Let** $X(G)$ **denote the number of short cycles (of length at most** k**) in a random graph** $G \in \mathcal{G}(n, p)$.
- $E(X) = \sum_{i=3}^{k} \frac{(n)_i p^i}{2i} \leq \frac{1}{2} \sum_{i=3}^{k} (n^i p^i) \leq \frac{1}{2}(k-2) n^k p^k$.
- $P[X \geq n/2] \leq \frac{E(X)}{n/2} \leq (k-2) n^{k-1} p^k = (k-2) n^{k\epsilon - 1}$.
- **Since** $k\epsilon - 1 < 0$, $\lim_{n \to \infty} P[X \geq n/2] = 0$.

- **Let $\mathcal{P}$ be a graph property- i.e. a class of graphs closed under isomorphism.**
- **Let $p = p(n)$ be a fixed function. If $P[G \in \mathcal{P}] \to 1$, as $n \to \infty$, we say that $G \in \mathcal{P}$ for almost all $G \in \mathcal{G}(n, p)$.**
- **If $P[G \in \mathcal{P}] \to 0$ as $n \to \infty$, we say that almost no $G \in \mathcal{G}(n, p)$ has property $\mathcal{P}$.**

For every constant $p \in (0, 1)$**, and every graph** H**, almost every** $G \in \mathcal{G}(n, p)$**, contains an induced copy of** H**.**

We call a real function $t = t(n)$ **with** $t(n) \neq 0$**, for all** n**, a threshold function for a graph property** $\mathcal{P}$**, if the following holds for all** $p = p(n)$**, and** $G \in \mathcal{G}(n, p)$**.**
$\lim_{n \to \infty}[G \in \mathcal{P}] = 0$**, if** $p/t \to 0$**, as** $n \to \infty$ **and** $= 1$ **if** $p/t \to 1$**, as** $n \to \infty$**.**

- **Consider a graph property of the form $\mathcal{P} = \{G : X(G) \geq 1\}$ where $X \geq 0$ is a random variable on $G(n, p)$. (Example, connectedness).**
- **How can we prove that $\mathcal{P}$ has a threshold function t ?**
- **We study one method here, called second moment method.**
- **If we can show that as $n \to \infty$, $E(X) \to 0$, then it means, that almost all graphs have property $\mathcal{P}$. (Since $P[X \geq 1] \leq E(X)$, by Markov inequality.)**
- **On the other hand we cannot show**

The Variance σ^2 of X: $\sigma^2 = E((X-\mu)^2)$. It is a quadratic measure of how much X deviates from its mean.

$$\sigma^2 = E(X^2) - \mu^2.$$

Chebyshev's Inequality: For all real $\lambda > 0$,
$P[|X - \mu| \geq \lambda] \leq \frac{\sigma^2}{\lambda^2}$**.**

If $\mu > 0$**, for** n **large, and** $\frac{\sigma^2}{\mu^2} \to 0$**, as** $n \to \infty$**, then** $X(G) > 0$
Since any graph G **with** $X(G) = 0$ **satisfies** $|X(G) - \mu| = \mu$**. So,**
$P[X = 0] \leq P[|X - \mu| \geq \mu] \leq \frac{\sigma^2}{\mu^2} \to 0$**, as** $n \to 0$**.**

Given a graph H, let $\mathcal{P}_H$ be the property of containing a copy of H as subgraph. H is called balanced if $\epsilon(H') \leq \epsilon(H)$ for all subgraphs H' of H.

If H is a balanced graph with k vertices, and $\ell \geq 1$ edges, then $t(n) = n^{-k/\ell}$ is a threshold function for $\mathcal{P}_H$.

If $k \geq 3$, then $t(n) = n^{-1}$ is a threshold function for the property of containing a k-cycle.

If T is a tree of order $k \geq 2$, then $t(n) = n^{k/(k-1)}$ is a threshold function for the property for containing a copy of T.

If $k \geq 2$**, then** $t(n) = n^{2/(k-1)}$ **is a threshold function for the property of containing a** K_k**.**

- **Let $X(G)$ denote the number of subgraphs of G isomorphic to H.**
- **Given $n \in N$, let $\mathcal{H}$ denote the set of all graphs isomorphic to H whose vertices lie in $\{0, 1, \ldots, n-1\}$.**
- **Given $H' \in \mathcal{H}$, we write $H' \subseteq G$ to denote that H' itself is a subgraph of G.**
- **The number of isomorphic copies of H on a fixed k set is at most $k!$.**
- $|\mathcal{H}| \leq \binom{n}{k} k! \leq n^k$.
- **Given $p = p(n)$, let $\gamma = p/t$, where $t = n^{-k/\ell}$.**

- **For each fixed** $H' \in \mathcal{H}$**,** $P[H' \subseteq G] = p^{\ell}$ **since** $|E(H')| = \ell$**.**
- $E(X) = |\mathcal{H}|p^{\ell} \leq n^{k}(\gamma n^{-k/\ell})^{\ell} = \gamma^{\ell} \to 0$**, if** $\gamma \to 0$ **as** $n \to 0$**.**

We have $\frac{\binom{n}{k}}{n^k} \geq \frac{1}{k!}\left(1 - \frac{k-1}{k}\right)^k$**.**