Graph Theory: Lecture No. 38

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Given a graph H, let $\mathcal{P}_H$ be the property of containing a copy of H as subgraph. H is called balanced if $\epsilon(H') \leq \epsilon(H)$ for all subgraphs H' of H.

If H is a balanced graph with k vertices, and $\ell \geq 1$ edges, then $t(n) = n^{-k/\ell}$ is a threshold function for $\mathcal{P}_H$.

- **Let $X(G)$ denote the number of subgraphs of G isomorphic to H.**
- **Given $n \in N$, let $\mathcal{H}$ denote the set of all graphs isomorphic to H whose vertices lie in $\{0, 1, \ldots, n-1\}$.**
- **Given $H' \in \mathcal{H}$, we write $H' \subseteq G$ to denote that H' itself is a subgraph of G.**
- **The number of isomorphic copies of H on a fixed k set is at most $k!$.**
- $|\mathcal{H}| \leq \binom{n}{k} k! \leq n^k$.
- **Given $p = p(n)$, let $\gamma = p/t$, where $t = n^{-k/\ell}$.**

- **For each fixed** $H' \in \mathcal{H}$**,** $P[H' \subseteq G] = p^\ell$ **since** $|E(H')| = \ell$**.**
- $E(X) = |\mathcal{H}|p^\ell \leq n^k(\gamma n^{-k/\ell})^\ell = \gamma^\ell \to 0$**, if** $\gamma \to 0$ **as** $n \to 0$**.**

If $\mu > 0$**, for** n **large, and** $\frac{\sigma^2}{\mu^2} \longrightarrow 0$**, as** $n \longrightarrow \infty$**, then** $X(G) > 0$
Since any graph G **with** $X(G) = 0$ **satisfies** $|X(G) - \mu| = \mu$**. So,**
$P[X = 0] \leq P[|X - \mu| \geq \mu] \leq \frac{\sigma^2}{\mu^2} \longrightarrow 0$**, as** $n \longrightarrow 0$**.**

We have $\frac{\binom{n}{k}}{n^k} \geq \frac{1}{k!}\left(1 - \frac{k-1}{k}\right)^k$**.**

$$\binom{n}{k} n^{-k} = \frac{1}{k!}\left(\frac{n}{n} \cdots \frac{n-k+1}{n}\right)$$

$$\geq \frac{1}{k!}\left(\frac{n-k+1}{n}\right)^k$$

$$\geq \frac{1}{k!}\left(1 - \frac{k-1}{k}\right)^k$$

We need to show that $\frac{\sigma^2}{\mu^2} = \frac{E(X^2)-\mu^2}{\mu^2} \longrightarrow 0$**, as** $\gamma \longrightarrow 0$**, i.e. as** $n \longrightarrow \infty$**.**

$$E(X^2) = \sum_{(H',H'')\in\mathcal{H}^2} P[H' \cup H'' \subseteq G]$$

$P[H' \cup H'' \subseteq G] = p^{2\ell - ||H' \cap H''||}$
Since H **is balanced** $||H' \cap H''|| \leq i\ell/k$**, if** $|H' \cap H''| = i$**. So,**
$P[H' \cup H'' \subseteq G] \leq p^{2\ell - i\ell/k}$

For $0 \leq i \leq k$**, let** $\mathcal{H}_i^2 = \{(H', H'') \in \mathcal{H}^2 : |H' \cap H''| = i\}$ **and** $A_i = \sum_i P[H' \cup H'' \subseteq G]$**, be the corresponding sum.**

For $i = 0$, H' and H'' are disjoint, and so the events $H' \subseteq G$ and $H'' \subseteq G$ are independent. Hence,

$A_0 = \sum_0 P[H' \cup H'' \subseteq G]$
$= \sum_0 P[H' \subseteq G]P[H'' \subseteq G]$
$\leq \sum_{(H', H'') \in \mathcal{H}^2} P[H' \subseteq G].P[H'' \subseteq G] \leq \mu^2$

For $i \geq 1$:

$$A_i = \sum_i P[H' \cup H'' \subset G]$$

$$\leq \sum{}' \binom{k}{i} \binom{n-k}{k-i} hp^{2l} p^{-il/k}$$

$$= |\mathcal{H}| \binom{k}{i} \binom{(n-k)}{(k-i)} hp^{2l} (\gamma n^{-k/l})^{-il/k}$$

$$\leq |\mathcal{H}| p^l c_1 n^{k-i} hp^l \gamma^{-il/k} n^i$$

$$= \mu c_1 n^k hp^\ell \gamma^{-i\ell/k}$$

$$\leq \mu c_2 \binom{n}{k} hp^\ell \gamma^{-i\ell/k}$$

$$\leq \mu^2 c_2 \gamma^{-i\ell/k} \leq \mu^2 c_2 \gamma^{-\ell/k}$$

- **An event** E **is mutually independent of the events** $E_1, E_2, \ldots, E_n$**, if for any subset** $I \subseteq [1, n]$**,** $P(E | \cap_{j \in I} E_j) = P(E)$
- **A dependency graph for a set of events** $E_1, E_2, \ldots, E_n$ **is a graph** $G = (V, E)$ **such that** $V = \{1, 2, \ldots, n\}$ **and for** $i = 1, \ldots, n$**, event** E_i **is mutually independent of the events** $\{E_j : (i, j) \notin E\}$**.**

Lovasz Local Lemma: Let $E_1, E_2, \ldots, E_n$ be a set of events and assume that the following hold:

1. **for all i, $P(E_i) \leq p$**
2. **The degree of the dependency graph given by $E_1, E_2, \ldots, E_n$ is bounded above by d**
3. $4dp \leq 1$

Then $P\left(\cap_{i=1}^{n} \overline{E_i}\right) > 0$.