Graph Theory: Lecture No. 39

L. Sunil Chandran

Computer Science and Automation,
Indian Institute of Science, Bangalore
Email: sunil@csa.iisc.ernet.in

1. Contract an edge.
2. Delete a vertex.
3. Delete an edge.

Let G be a graph. If X is another graph, and $\{V_x : x \in V(X)\}$ is a partition of $V(G)$ into connected subsets such that for any two vertices $x, y \in X$, there is a V_x-V_y edge in G if and only if $(x, y) \in E(G)$, we say that G is an MX and write $G = MX$.
V_x are the branch sets of MX

G **is an** MX **if and only if** X **can be obtained from** G **by a series of edge contractions, i.e. if and only if there are graphs** $G_0, \ldots, G_n$ **and edges** $e_i \in G_i$ **such that** $G_0 = G$**,** $G_n = X$ **and** $G_{i+1} = G_i/e_i$**, for all** $i < n$**.**

If $G = MX$ is a subgraph of another graph Y, then we say that X is a minor of Y.

If we replace the edges of X with independent paths between ends, we call the graph G obtained a subdivision of X, and write $G = TX$. If TX is a subgraph of Y then X is a topological minor of Y.

If $\Delta(X) \leq 3$ **then every** MX **contains a** TX**.**

Hadwiger's Conjecture: The following implication holds for every integer $r > 0$ and every graph G.
$\chi(G) \geq r$ **implies that** K_r **is a minor of** G

A graph with at least 3 vertices is edge maximal without a K_4 minor if and only if it can be constructed recursively from triangles by pasting along K_2s.

Every edge maximal graph without a K_4 minor has $2|G| - 3$ edges.

Hadwiger's Conjecture holds for $r = 4$

Wagner, 1937: Let G be an edge maximal graph without a K_5 minor. If $|G| \geq 4$, then G can be constructed recursively, by pasting along K_3s and K_2s from plane triangulations and copies of the graph W.

A graph with n vertices and no K_5 minor has at most $3n - 6$ edges.

Hadwiger's conjecture holds for $r = 5$.

(Robertson, Seymour and Thomas, 1993)
Hadwiger's conjecture holds for $r = 6$.

Kühn and Osthus: For every integer s, there is an integer r_s such that Hadwiger Conjecture holds for all graphs $G \not\supseteq K_{s,s}$ and $r \geq r_s$.

There is a constant g such that all graphs G of girth at least g satisfy the implication $\chi(G) \geq r \rightarrow G \supseteq TK_r$ for all r.

There is a constant $c \in R$ such that for $r \in N$, every graph G of average degree $d(G) \geq cr^2$ contains K_r as a topological minor.

Kostochka, 1982: There exists a constant $c \in R$ such that for every $r \in N$, every graph G of average degree $d(G) \geq cr\sqrt{\log r}$ contains K_r as a minor. Up to the value of c, this bound is best possible as a function of r.

Let $d, k \in N$ **with** $d \geq 3$ **and let** G **be a graph of minimum degree** $\delta(G) \geq d$ **and girth** $g(G) \geq 8k + 3$**. Then** G **has a minor** H **of minimum degree** $\delta(H) \geq d(d-1)^k$**.**

Thomassen, 1983: There exists a function $f : N \longrightarrow N$, such that every graph of minimum degree at least 3 and girth at least $f(r)$ has a K_r minor, for all $r \in N$.

Take $f(r) = 8\log r + 4\log\log r + c$ **for some constant** $c \in R$**. Take** $k = k(r)$ **minimal with** $3.2^k \geq c' r\sqrt{\log r}$**, where** c' **is the constant from Kostochka's Lemma.**

There exists a constant g such that $G \supseteq TK_r$ for every graph G satisfying $\delta(G) \geq r - 1$ and girth $\geq g$.