Numerical Optimization

Constrained Optimization - Algorithms

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NPTEL Course on Numerical Optimization

Some Optimization Formulations

Let $\boldsymbol{H}$ be a symmetric positive definite matrix.

$$\begin{array}{rl} \min & \frac{1}{2}\boldsymbol{x}^T\boldsymbol{H}\boldsymbol{x} + \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{a}_i^T\boldsymbol{x} = b_i, \quad i \in \mathcal{E} \end{array}$$

$$\begin{array}{rl} \min & \frac{1}{2}\boldsymbol{x}^T\boldsymbol{H}\boldsymbol{x} + \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{a}_i^T\boldsymbol{x} = b_i, \quad i \in \mathcal{E} \\ & \boldsymbol{a}_i^T\boldsymbol{x} \leq b_i, \quad i \in \mathcal{I} \end{array}$$

$$\begin{array}{rl} \min & f(\boldsymbol{x}) \\ \text{s.t.} & \boldsymbol{a}_i^T\boldsymbol{x} = b_i, \quad i \in \mathcal{E} \\ & \boldsymbol{a}_i^T\boldsymbol{x} \leq b_i, \quad i \in \mathcal{I} \end{array}$$

$$\begin{array}{rl} \min & f(\boldsymbol{x}) \\ \text{s.t.} & h_j(\boldsymbol{x}) \leq 0, \; j = 1, \ldots, l \\ & e_i(\boldsymbol{x}) = 0, \; i = 1, \ldots, m \end{array}$$

Some Optimization Methods

- Lagrange Methods
- Penalty and Barrier Function Methods
- Cutting-Plane Methods

Lagrange Methods

- Quadratic Program with Linear Equality Constraints

$$\begin{array}{rl} \min & \frac{1}{2}\boldsymbol{x}^T\boldsymbol{H}\boldsymbol{x} + \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{a}_i^T\boldsymbol{x} = b_i, \quad i \in \mathcal{E} \end{array}$$

where $\boldsymbol{H}$ is a symmetric positive definite matrix.

or

$$\begin{array}{rl} \min & \frac{1}{2}\boldsymbol{x}^T\boldsymbol{H}\boldsymbol{x} + \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \end{array}$$

where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ and $rank(\boldsymbol{A}) = m$.
First order necessary and sufficient conditions:

$$\left.\begin{array}{r} \boldsymbol{H}\boldsymbol{x} + \boldsymbol{A}^T\boldsymbol{\lambda} + \boldsymbol{c} = 0 \\ \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \end{array}\right\} (n+m) \text{ equations in } (n+m) \text{ unknowns}$$

$$
\begin{aligned}
\boldsymbol{H}\boldsymbol{x} + \boldsymbol{A}^T\boldsymbol{\lambda} + \boldsymbol{c} &= 0 \\
\boldsymbol{A}\boldsymbol{x} &= \boldsymbol{b}
\end{aligned}
$$

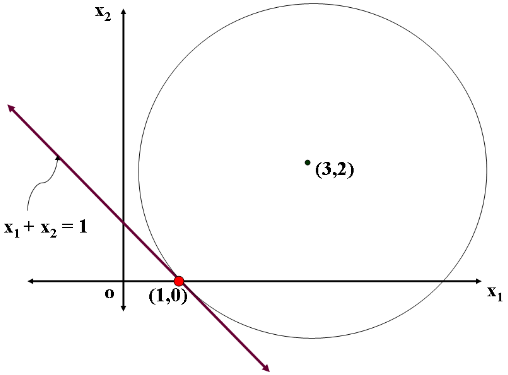
$$
\begin{aligned}
\therefore \boldsymbol{x} &= -\boldsymbol{H}^{-1}(\boldsymbol{A}^T\boldsymbol{\lambda} + \boldsymbol{c}) \\
\therefore \; -\boldsymbol{A}\boldsymbol{H}^{-1}(\boldsymbol{A}^T\boldsymbol{\lambda} + \boldsymbol{c}) &= \boldsymbol{b} \\
\therefore \boldsymbol{\lambda} = -(\boldsymbol{A}\boldsymbol{H}^{-1}\boldsymbol{A}^T)^{-1}&(\boldsymbol{A}\boldsymbol{H}^{-1}\boldsymbol{c} + \boldsymbol{b})
\end{aligned}
$$

Using this value of $\boldsymbol{\lambda}$,

$$
\boldsymbol{x} = -\boldsymbol{H}^{-1}(\boldsymbol{I} - \boldsymbol{A}^T(\boldsymbol{A}\boldsymbol{H}^{-1}\boldsymbol{A}^T)^{-1}\boldsymbol{A}\boldsymbol{H}^{-1})\boldsymbol{c} + \boldsymbol{H}^{-1}\boldsymbol{A}^T(\boldsymbol{A}\boldsymbol{H}^{-1}\boldsymbol{A}^T)^{-1}\boldsymbol{b}
$$

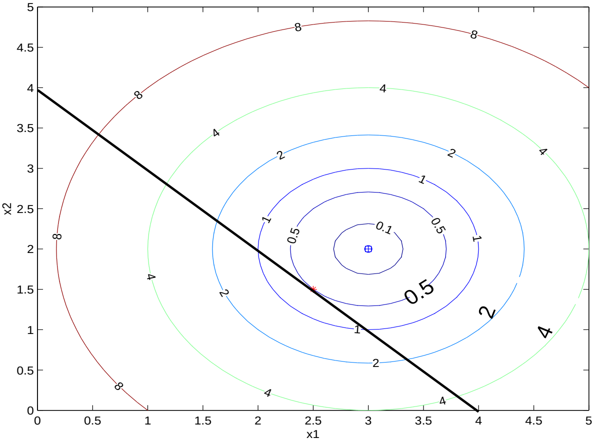
Example:

$$\begin{array}{ll} \min & \frac{1}{2}[(x_1 - 3)^2 + (x_2 - 2)^2] \\ \text{s.t.} & x_1 + x_2 = 1 \end{array}$$



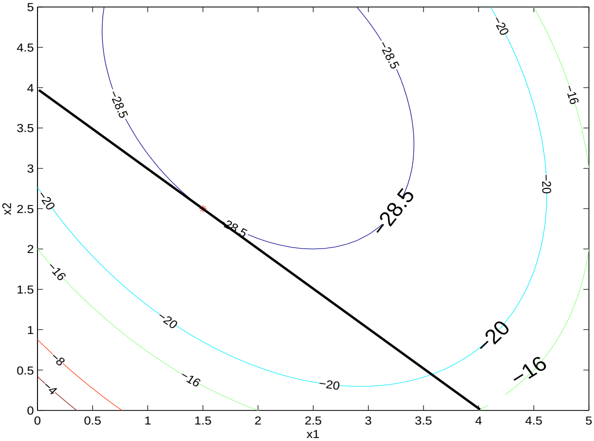
Example:

$$\begin{array}{ll} \min & \frac{1}{2}[(x_1-3)^2+(x_2-2)^2] \\ \text{s.t.} & x_1+x_2=4 \end{array}$$



Example:

$$\begin{array}{ll} \min & 2x_1^2 + x_1x_2 + x_2^2 - 12x_1 - 10x_2 \\ \text{s.t.} & x_1 + x_2 = 4 \end{array}$$



- Quadratic Program with Linear Equality and Inequality Constraints

$$\begin{aligned} \min \quad & \frac{1}{2}\boldsymbol{x}^T\boldsymbol{H}\boldsymbol{x} + \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{a}_i^T\boldsymbol{x} = b_i, \quad i \in \mathcal{E} \\ & \boldsymbol{a}_i^T\boldsymbol{x} \leq b_i, \quad i \in \mathcal{I} \end{aligned}$$

where $\boldsymbol{H}$ is a symmetric positive definite matrix.

Each step of an active set algorithm:

- Given $\boldsymbol{x}^k$, a feasible point at $k-$th iteration, define the *working set*, W^k as,

$$W^k = \mathcal{E} \cup \{i \in \mathcal{I} : \boldsymbol{a}_i^T\boldsymbol{x} = b_i\}$$

- Find a descent direction, $\boldsymbol{d}^k$, w.r.t. W^k
- $\boldsymbol{x}^{k+1} = \boldsymbol{x}^k + \alpha^k\boldsymbol{d}^k$ where $\alpha^k > 0$

Given a feasible point $\boldsymbol{x}^k$ and W^k, find $\boldsymbol{d}^k$ by solving the problem:

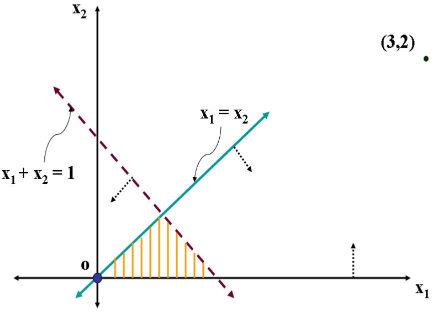
$$\begin{aligned} \min_{\boldsymbol{d}} \quad & \frac{1}{2}\boldsymbol{d}^T\boldsymbol{H}\boldsymbol{d} + \boldsymbol{g}^{k^T}\boldsymbol{d} \\ \text{s.t.} \quad & \boldsymbol{a}_i^T\boldsymbol{d} = 0, \quad i \in W^k \end{aligned}$$

where $\boldsymbol{g}^k = \boldsymbol{H}\boldsymbol{x}^k + \boldsymbol{c}$.

- If $\boldsymbol{d}^k = \boldsymbol{0}$
  - $\boldsymbol{x}^k$ is optimal w.r.t. W^k
  - Check if $\lambda_i \geq 0, \quad i \in \mathcal{I} \cap W^k$
  - Drop a constraint, if necessary, to form W^{k+1}
- If $\boldsymbol{d}^k \neq \boldsymbol{0}$
  - Find the step length α^k such that $\boldsymbol{x}^{k+1}$ is feasible w.r.t. $\mathcal{E} \cup \mathcal{I}$
  - Add a constraint, if necessary, to form W^{k+1}

Example:

$$\begin{array}{ll} \min & \frac{1}{2}[(x_1-3)^2+(x_2-2)^2] \\ \text{s.t.} & -x_1+x_2 \le 0 \\ & x_1+x_2 \le 1 \\ & -x_2 \le 0 \end{array}$$

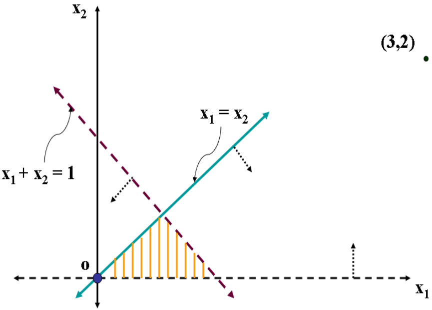


- $\boldsymbol{H} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\boldsymbol{c} = (-3,-2)^T$

(1) Let $\boldsymbol{x}^0 = \mathbf{0}$.

- $\boldsymbol{g}^0 = \boldsymbol{H}\boldsymbol{x}^0 + \boldsymbol{c} = \boldsymbol{c}$
- $W^0 = \{1,3\}$
- Solution of Quadratic Program associated with W^0: $\boldsymbol{d}^0 = \mathbf{0}$
- $\boldsymbol{\lambda} = (-3,-5)^T$
- Suppose $W^1 = \{1\}$ (constraint 3 is dropped)

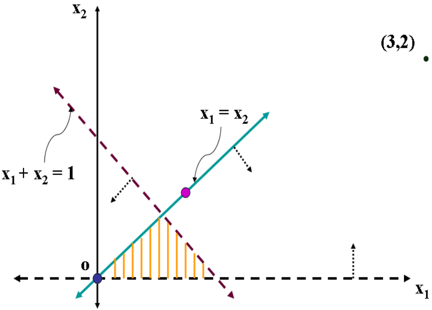
$$
\begin{array}{ll}
\min & \frac{1}{2}[(x_1-3)^2+(x_2-2)^2] \\
\text{s.t.} & -x_1+x_2 \leq 0 \\
& x_1+x_2 \leq 1 \\
& -x_2 \leq 0
\end{array}
$$



(2) $\boldsymbol{x}^1 = \boldsymbol{0}$.

- $\boldsymbol{g}^1 = \boldsymbol{H}\boldsymbol{x}^0 + \boldsymbol{c} = \boldsymbol{c}$
- $W^1 = \{1\}$
- Solution of Quadratic Program associated with W^1: $\boldsymbol{d}^1 = (\frac{5}{2}, \frac{5}{2})^T$

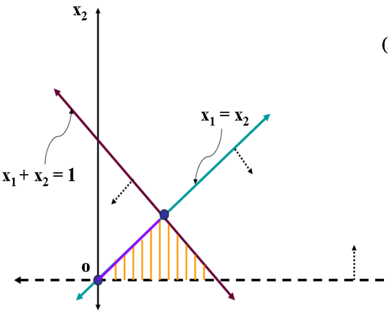
$$\begin{array}{ll} \min & \frac{1}{2}[(x_1-3)^2+(x_2-2)^2] \\ \text{s.t.} & -x_1+x_2 \le 0 \\ & x_1+x_2 \le 1 \\ & -x_2 \le 0 \end{array}$$



(2) $\boldsymbol{x}^1 = \boldsymbol{0}$.

- $\boldsymbol{g}^1 = \boldsymbol{H}\boldsymbol{x}^0 + \boldsymbol{c} = \boldsymbol{c}$
- $W^1 = \{1\}$
- Solution of Quadratic Program associated with W^1: $\boldsymbol{d}^1 = (\frac{5}{2}, \frac{5}{2})^T$
- $\alpha^1 = 1 \;\Rightarrow\; \boldsymbol{x}^2 = (\frac{5}{2}, \frac{5}{2})^T$ (not feasible)
- $\alpha^1 = \frac{1}{5} \;\Rightarrow\; \boldsymbol{x}^2 = (\frac{1}{2}, \frac{1}{2})^T$
- $W^2 = \{1, 2\}$ (constraint 2 is added)

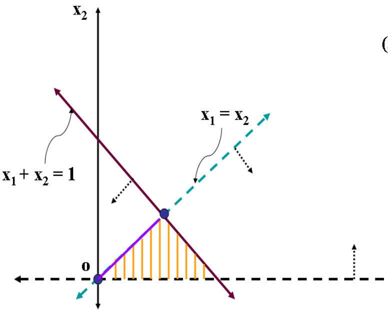
$$\begin{array}{ll} \min & \frac{1}{2}[(x_1-3)^2+(x_2-2)^2] \\ \text{s.t.} & -x_1+x_2 \le 0 \\ & x_1+x_2 \le 1 \\ & -x_2 \le 0 \end{array}$$



(3) $\boldsymbol{x}^2 = (\frac{1}{2}, \frac{1}{2})^T$.

- $\boldsymbol{g}^2 = \boldsymbol{H}\boldsymbol{x}^2 + \boldsymbol{c} = (-\frac{5}{2}, -\frac{3}{2})^T$
- $W^2 = \{1, 2\}$
- Solution of Quadratic Program associated with W^2: $\boldsymbol{d}^2 = \boldsymbol{0}$
- $\boldsymbol{\lambda} = (-\frac{1}{2}, 2)^T$
- $W^2 = \{2\}$ (constraint 1 is removed)

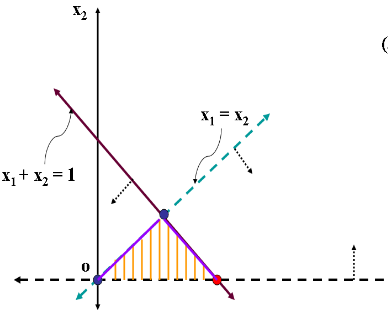
$$\begin{array}{ll} \min & \frac{1}{2}[(x_1-3)^2+(x_2-2)^2] \\ \text{s.t.} & -x_1+x_2 \leq 0 \\ & x_1+x_2 \leq 1 \\ & -x_2 \leq 0 \end{array}$$



(4) $\boldsymbol{x}^3 = (\frac{1}{2}, \frac{1}{2})^T$.

- $\boldsymbol{g}^3 = \boldsymbol{H}\boldsymbol{x}^3 + \boldsymbol{c} = (-\frac{5}{2}, -\frac{3}{2})^T$
- $W^3 = \{2\}$
- Solution of Quadratic Program associated with W^3: $\boldsymbol{d}^3 = (\frac{1}{2}, -\frac{1}{2})^T$
- $\boldsymbol{x}^4 = \boldsymbol{x}^3 + \boldsymbol{d}^3 = (1, 0)^T$ (feasible point)

$$\begin{array}{ll} \min & \frac{1}{2}[(x_1-3)^2+(x_2-2)^2] \\ \text{s.t.} & -x_1+x_2 \le 0 \\ & x_1+x_2 \le 1 \\ & -x_2 \le 0 \end{array}$$



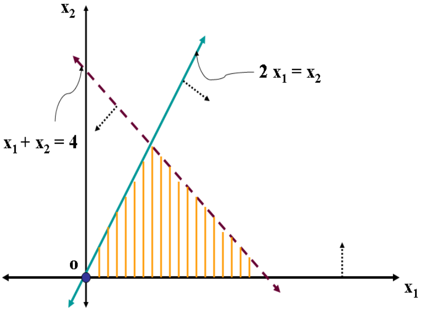
(4) $\boldsymbol{x}^3 = (\frac{1}{2}, \frac{1}{2})^T$.

- $\boldsymbol{g}^3 = \boldsymbol{H}\boldsymbol{x}^3 + \boldsymbol{c} = (-\frac{5}{2}, -\frac{3}{2})^T$
- $W^3 = \{2\}$
- Solution of Quadratic Program associated with W^3: $\boldsymbol{d}^3 = (\frac{1}{2}, -\frac{1}{2})^T$
- $\boldsymbol{x}^4 = \boldsymbol{x}^3 + \boldsymbol{d}^3 = (1,0)^T$ (feasible point)
- $\boldsymbol{\lambda} = 2$

$\boldsymbol{x}^* = (1,0)^T$

Example:

$$
\begin{array}{ll}
\min & \frac{1}{2}x_1^2 + x_2^2 - 3x_1 - 4x_2 \\
\text{s.t.} & -2x_1 + x_2 \leq 0 \\
& x_1 + x_2 \leq 4 \\
& -x_2 \leq 0
\end{array}
$$

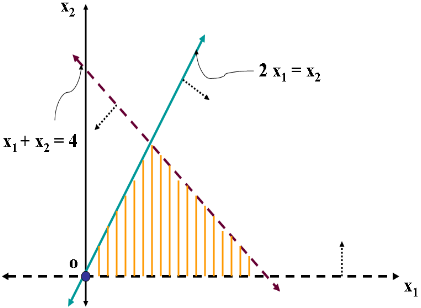


- $\boldsymbol{H} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $\boldsymbol{c} = (-3, -4)^T$

(1) Let $\boldsymbol{x}^0 = \mathbf{0}$.

- $\boldsymbol{g}^0 = \boldsymbol{H}\boldsymbol{x}^0 + \boldsymbol{c} = \boldsymbol{c}$
- $W^0 = \{1, 3\}$
- Solution of Quadratic Program associated with W^0: $\boldsymbol{d}^0 = \mathbf{0}$
- $\boldsymbol{\lambda} = (-\frac{3}{2}, -\frac{11}{2})^T$
- Suppose $W^1 = \{1\}$ (constraint 3 is dropped)

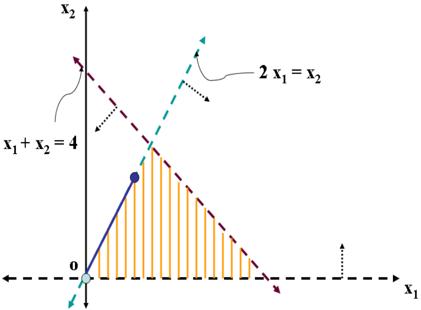
$$
\begin{array}{ll}
\min & \frac{1}{2}x_1^2 + x_2^2 - 3x_1 - 4x_2 \\
\text{s.t.} & -2x_1 + x_2 \leq 0 \\
& x_1 + x_2 \leq 4 \\
& -x_2 \leq 0
\end{array}
$$



(2) $\boldsymbol{x}^1 = \boldsymbol{0}$.

- $\boldsymbol{g}^1 = \boldsymbol{H}\boldsymbol{x}^0 + \boldsymbol{c} = \boldsymbol{c}$
- $W^1 = \{1\}$
- Solution of Quadratic Program associated with W^1: $\boldsymbol{d}^1 = (\frac{11}{9}, \frac{22}{9})^T$
- $\alpha^1 = 1 \Rightarrow \boldsymbol{x}^2 = (\frac{11}{9}, \frac{22}{9})^T$ (feasible)
- $\lambda = -\frac{8}{9}$
- $W^2 = \phi$ (constraint 1 is removed)

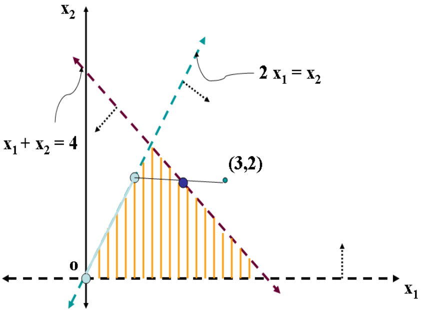
$$\begin{array}{ll} \min & \frac{1}{2}x_1^2 + x_2^2 - 3x_1 - 4x_2 \\ \text{s.t.} & -2x_1 + x_2 \le 0 \\ & x_1 + x_2 \le 4 \\ & -x_2 \le 0 \end{array}$$



(3) $\boldsymbol{x}^2 = (\frac{11}{9}, \frac{22}{9})^T$.

- $W^2 = \phi$
- Solution of Quadratic Program (unconstrained) $\boldsymbol{x}^3 = (3, 2)^T$ (not feasible)

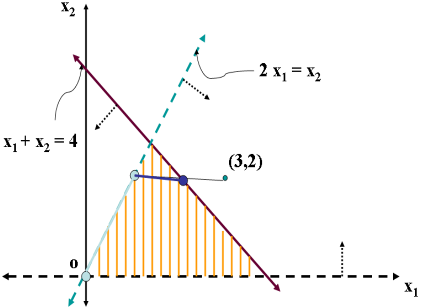
$$\begin{array}{ll} \min & \frac{1}{2}x_1^2 + x_2^2 - 3x_1 - 4x_2 \\ \text{s.t.} & -2x_1 + x_2 \le 0 \\ & x_1 + x_2 \le 4 \\ & -x_2 \le 0 \end{array}$$



(3) $\boldsymbol{x}^2 = (\frac{11}{9}, \frac{22}{9})^T$.

- $W^2 = \phi$
- Solution of Quadratic Program (unconstrained) $\boldsymbol{x}^3 = (3, 2)^T$ (not feasible)
- $\boldsymbol{d}^2 = (\frac{16}{9}, \frac{-4}{9})^T$
- $\alpha^2 = \frac{1}{4} \Rightarrow \boldsymbol{x}^3 = (\frac{5}{3}, \frac{7}{3})^T$ (feasible)
- $W^2 = \{2\}$ (constraint 2 is added)

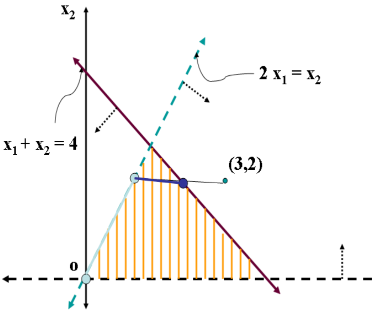
$$
\begin{array}{ll}
\min & \frac{1}{2}x_1^2 + x_2^2 - 3x_1 - 4x_2 \\
\text{s.t.} & -2x_1 + x_2 \leq 0 \\
& x_1 + x_2 \leq 4 \\
& -x_2 \leq 0
\end{array}
$$



(3) $\boldsymbol{x}^2 = (\frac{11}{9}, \frac{22}{9})^T$.

- $W^2 = \phi$
- Solution of Quadratic Program (unconstrained)
  $\boldsymbol{x}^3 = (3, 2)^T$ (not feasible)
- $\boldsymbol{d}^2 = (\frac{16}{9}, \frac{-4}{9})^T$
- $\alpha^2 = \frac{1}{4} \Rightarrow \boldsymbol{x}^3 = (\frac{5}{3}, \frac{7}{3})^T$ (feasible)
- $W^2 = \{2\}$ (constraint 2 is added)

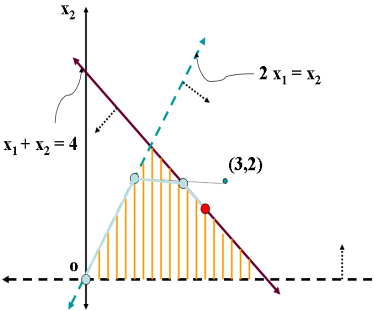
$$\begin{array}{ll} \min & \frac{1}{2}x_1^2 + x_2^2 - 3x_1 - 4x_2 \\ \text{s.t.} & -2x_1 + x_2 \le 0 \\ & x_1 + x_2 \le 4 \\ & -x_2 \le 0 \end{array}$$



(4) $\boldsymbol{x}^3 = (\frac{5}{3}, \frac{7}{3})^T$.

- $\boldsymbol{g}^3 = \boldsymbol{H}\boldsymbol{x}^3 + \boldsymbol{c} = (-\frac{4}{3}, \frac{2}{3})^T$
- $W^3 = \{2\}$
- Solution of Quadratic Program associated with W^3:
  $\boldsymbol{d}^3 = (\frac{2}{3}, -\frac{2}{3})^T$
- $\boldsymbol{x}^4 = \boldsymbol{x}^3 + \boldsymbol{d}^3 = (\frac{7}{3}, \frac{5}{3})^T$ (feasible point)

$$\begin{aligned} \min \quad & \tfrac{1}{2}x_1^2 + x_2^2 - 3x_1 - 4x_2 \\ \text{s.t.} \quad & -2x_1 + x_2 \le 0 \\ & x_1 + x_2 \le 4 \\ & -x_2 \le 0 \end{aligned}$$



(4) $\boldsymbol{x}^3 = (\frac{5}{3}, \frac{7}{3})^T$.

- $\boldsymbol{g}^3 = \boldsymbol{H}\boldsymbol{x}^3 + \boldsymbol{c} = (-\frac{4}{3}, \frac{2}{3})^T$
- $W^3 = \{2\}$
- Solution of Quadratic Program associated with W^3:
  $\boldsymbol{d}^3 = (\frac{2}{3}, -\frac{2}{3})^T$
- $\boldsymbol{x}^4 = \boldsymbol{x}^3 + \boldsymbol{d}^3 = (\frac{7}{3}, \frac{5}{3})^T$ (feasible point)
- $\lambda = \frac{2}{3}$

$\boldsymbol{x}^* = (\frac{7}{3}, \frac{5}{3})^T$

- Quadratic Program with Linear Equality and Inequality Constraints (**QP-LC**)

$$\begin{array}{rl} \min & \frac{1}{2}\boldsymbol{x}^T\boldsymbol{H}\boldsymbol{x} + \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{a}_i^T\boldsymbol{x} = b_i, \;\; i \in \mathcal{E} \\ & \boldsymbol{a}_i^T\boldsymbol{x} \leq b_i, \;\; i \in \mathcal{I} \end{array}$$

where $\boldsymbol{H}$ is a symmetric positive definite matrix.

Given a feasible point $\boldsymbol{x}^k$ and W^k, find $\boldsymbol{d}^k$ by solving the problem (**QP-SUB**):

$$\begin{array}{ll} \min_{\boldsymbol{d}} & \frac{1}{2}\boldsymbol{d}^T\boldsymbol{H}\boldsymbol{d} + {\boldsymbol{g}^k}^T\boldsymbol{d} \\ \text{s.t.} & \boldsymbol{a}_i^T\boldsymbol{d} = 0, \;\; i \in W^k \end{array} \equiv \begin{array}{ll} \min_{\boldsymbol{d}} & \frac{1}{2}\boldsymbol{d}^T\boldsymbol{H}\boldsymbol{d} + {\boldsymbol{g}^k}^T\boldsymbol{d} \\ \text{s.t.} & \boldsymbol{A}_{W^k}\boldsymbol{d} = \boldsymbol{0} \end{array}$$

where $\boldsymbol{g}^k = \boldsymbol{H}\boldsymbol{x}^k + \boldsymbol{c}$ and $\boldsymbol{A}_{W^k}^T = (\ldots, \boldsymbol{a}_i, \ldots), i \in W^k$.

Active Set Method (to solve **QP-LC**)

(1) Input: $\boldsymbol{H}, \boldsymbol{c}, \mathcal{E}, \mathcal{I}$

(2) Initialize $\boldsymbol{x}^0, W^0$, set $k = 0, StopFlag = 0$

(3) **while** $(StopFlag \neq 1)$

  (a) Find $\boldsymbol{A}_{W^k}$ and solve the corresponding **QP-SUB** to get $\boldsymbol{d}^k$

  (b) **if** $\boldsymbol{d}^k == \boldsymbol{0}$

  - $\boldsymbol{\lambda} = -(\boldsymbol{A}\boldsymbol{H}^{-1}\boldsymbol{A}^T)^{-1}(\boldsymbol{A}\boldsymbol{H}^{-1}\boldsymbol{c} + \boldsymbol{b})$
  - $\hat{\mathcal{I}} = \mathcal{I} \cap W^k, \lambda_q = \min_{i \in \hat{\mathcal{I}}} \lambda_i$
  - **if** $\lambda_q \geq 0$, set $StopFlag = 1$ **else** $W^{k+1} = W^k \setminus \{q\}$

  else

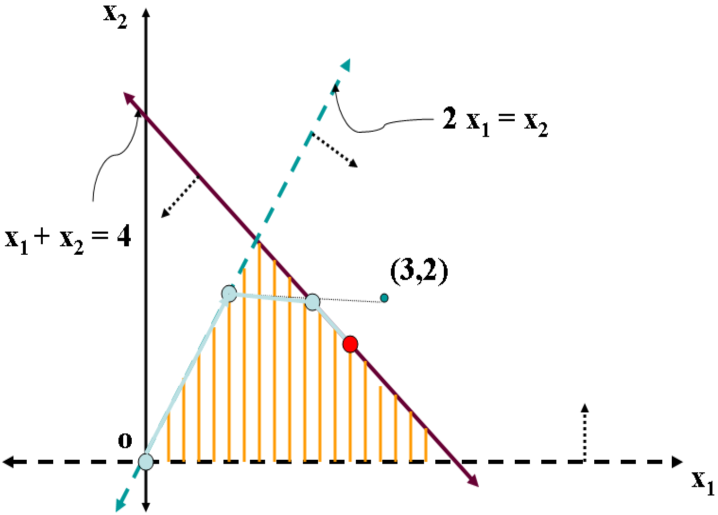
  - $temp = \min_{i: a_i^T d^k > 0} \left(\frac{b_i - a_i^T x^k}{a_i^T d^k}\right), p = \text{argmin}_{i: a_i^T d^k > 0} \left(\frac{b_i - a_i^T x^k}{a_i^T d^k}\right)$
  - $\alpha^k = \min(temp, 1), \boldsymbol{x}^{k+1} = \boldsymbol{x}^k + \alpha^k \boldsymbol{d}^k$
  - **if** $temp < 1, W^{k+1} = W^k \cup \{p\}$

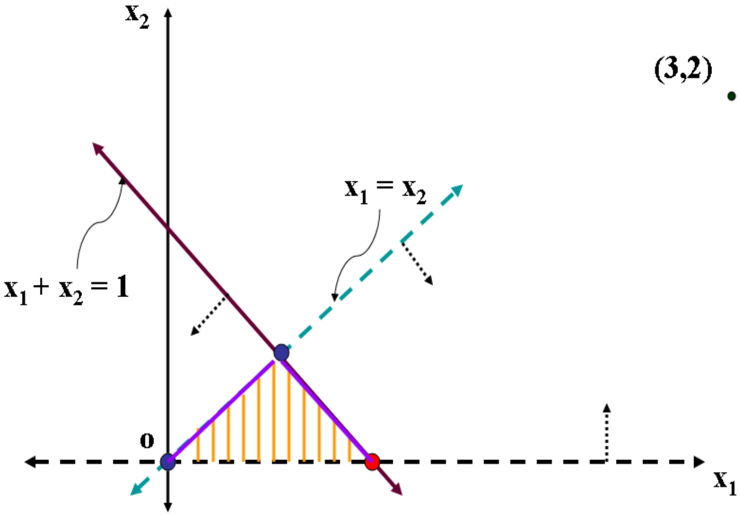
  endif

  (c) **if** $StopFlag == 0$, set $k := k + 1$ **endif**

endwhile

Output : $\boldsymbol{x}^* = \boldsymbol{x}^k$


x_2
$2\ x_1 = x_2$
$x_1 + x_2 = 4$
(3,2)
0
x_1


x2
(3,2)
x1 = x2
x1 + x2 = 1
0
x1