Numerical Optimization

Linear Programming

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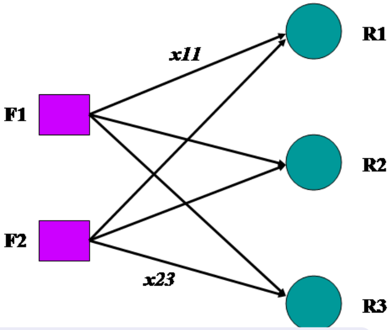
NPTEL Course on Numerical Optimization

Transportation Problem



$$\begin{array}{ll} \min_x & \sum_{ij} c_{ij} x_{ij} \\ \text{s.t.} & \sum_{j=1}^{3} x_{ij} \leq a_i, \ i = 1,2 \\ & \sum_{i=1}^{2} x_{ij} \geq b_j, \ j = 1,2,3 \\ & x_{ij} \geq 0 \ \forall \ i,j \end{array}$$

- a_i : Capacity of the plant *Fi*
- b_j : Demand of the outlet *Rj*
- c_{ij} : Cost of shipping one unit of product from *Fi* to *Rj*
- x_{ij}: Number of units of the product shipped from *Fi* to *Rj* (variables)
- The objective is to *minimize* $\sum_{ij} c_{ij} x_{ij}$
- $\sum_{j=1}^{3} x_{ij} \leq a_i, \ i = 1,2$ (constraints)
- $\sum_{i=1}^{2} x_{ij} \geq b_j, \ j = 1,2,3$ (constraints)
- $x_{ij} \geq 0 \ \forall \ i,j$ (constraints)

The Diet Problem: Find the *most economical* diet that satisfies *minimum* nutritional requirements.

- Number of food items: n
- Number of nutritional ingredient: m
- Each person must consume *at least* b_j units of nutrient j per day
- Unit cost of food item i: c_i
- Each unit of food item i contains a_{ji} units of the nutrient j
- Number of units of food item i consumed: x_i

Constraint corresponding to the nutrient j:

$$a_{j1}x_1 + a_{j2}x_2 + \ldots + a_{jn}x_n \geq b_j, \quad x_i \geq 0 \ \forall\ i$$

Cost:

$$c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

Problem:

$$\begin{aligned} \min \quad & c_1x_1 + c_2x_2 + \ldots + c_nx_n \\ \text{s.t.} \quad & a_{j1}x_1 + a_{j2}x_2 + \ldots + a_{jn}x_n \geq b_j \ \forall\, j \\ & x_i \geq 0 \ \forall\, i \end{aligned}$$

Given: $\boldsymbol{c} = (c_1, \ldots, c_n)^T, \boldsymbol{A} = (\boldsymbol{a}_1 | \ldots | \boldsymbol{a}_n), \boldsymbol{b} = (b_1, \ldots, b_m)^T$.
Linear Programming Problem (**LP**):

$$\begin{aligned} \min \quad & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

where $\boldsymbol{A} \in \mathbb{R}^{m \times n}, \boldsymbol{c} \in \mathbb{R}^n$ and $\boldsymbol{b} \in \mathbb{R}^m$.

- Assumption: $m \leq n, \text{rank}(\boldsymbol{A}) = m$
- Linear Constraints can be of the form $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ or $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$

Constraint (Feasible) Set:

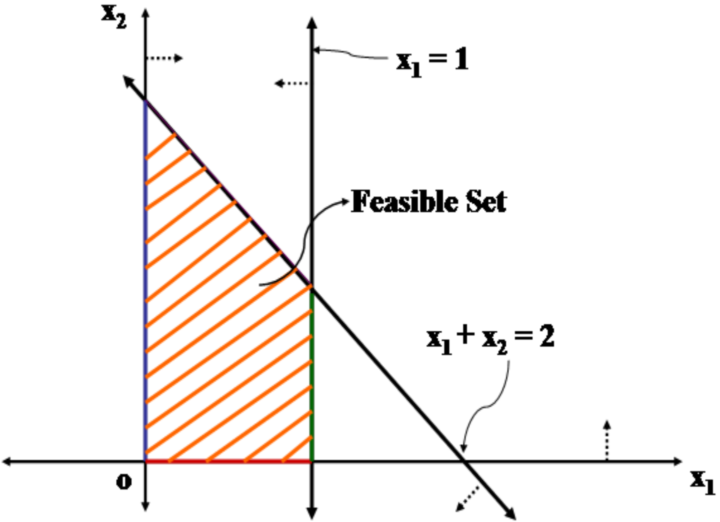
- Inequality constraint of the type $\{\boldsymbol{x} : \boldsymbol{a}^T\boldsymbol{x} \leq \boldsymbol{b}\}$ or $\{\boldsymbol{x} : \boldsymbol{a}^T\boldsymbol{x} \geq \boldsymbol{b}\}$ denotes a *half space*
- Equality constraint, $\{\boldsymbol{x} : \boldsymbol{a}^T\boldsymbol{x} = \boldsymbol{b}\}$, represents an affine space
- Non-negativity constraint, $\boldsymbol{x} \geq \boldsymbol{0}$
- Constraint set of an LP is a *convex* set

Polyhedral Set

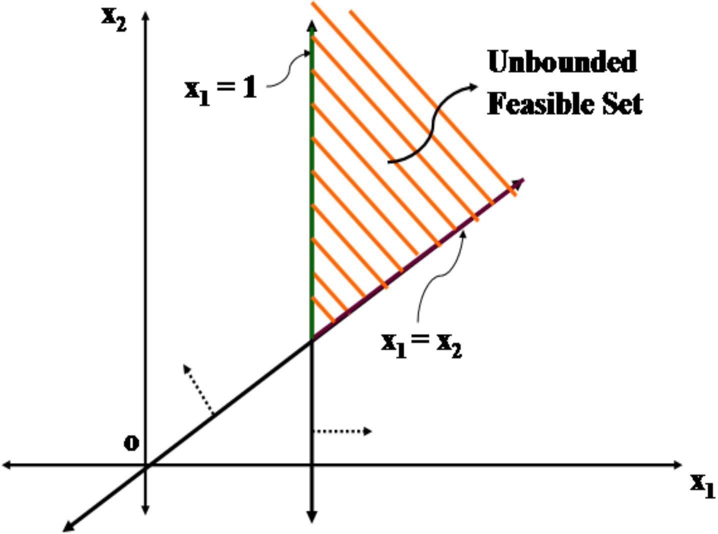
$$X = \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}\}$$

Polytope: A bounded polyhedral set

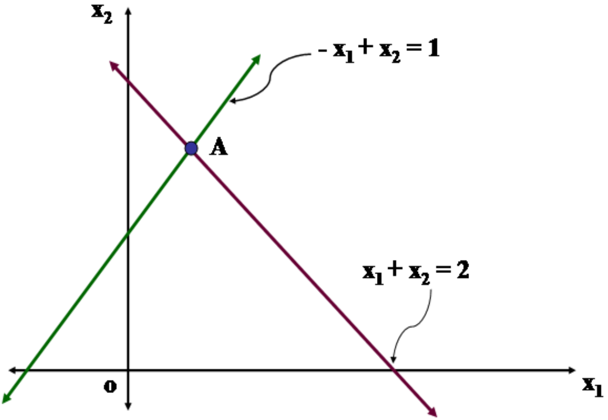
Consider the constraint set in $\mathbb{R}^2$:



Consider the constraint set in $\mathbb{R}^2$:

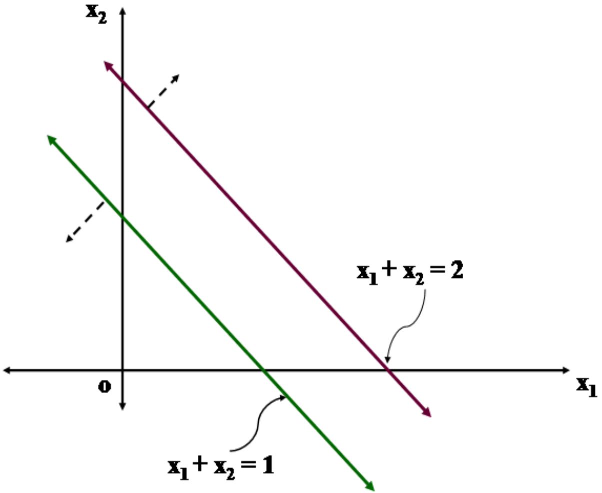


Feasible set can be a singleton set



Feasible Set = $\{(x_1, x_2) : x_1 + x_2 = 2, -x_1 + x_2 = 1\} = \{A\}$

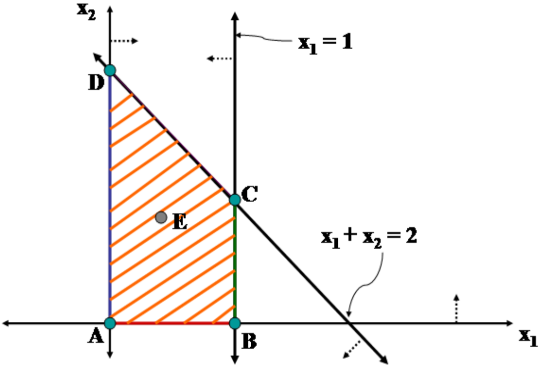
Feasible set can be empty!



Feasible Set = $\{(x_1, x_2) : x_1 + x_2 \geq 2, x_1 + x_2 \leq 1\} = \phi$

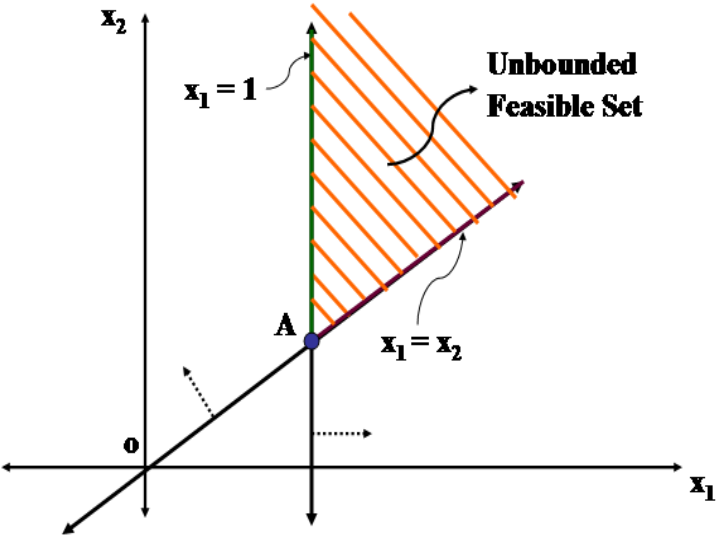
Definition

Let X be a convex set. A point $\boldsymbol{x} \in X$ is said to be an extreme point (corner point or vertex) of X if $\boldsymbol{x}$ cannot be represented as a strict convex combination of two distinct points in X.



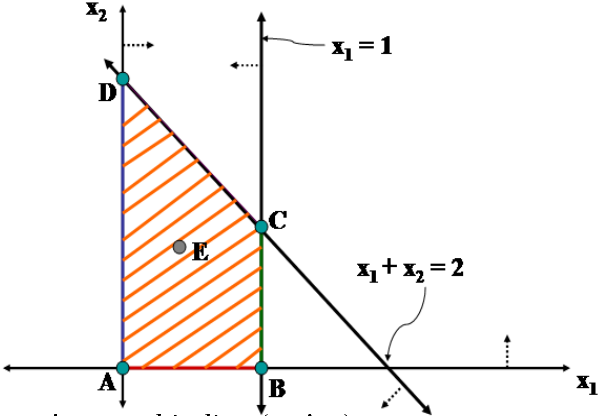
Extreme Points: A, B, C and D.

E is not an extreme point.



Extreme Point: A

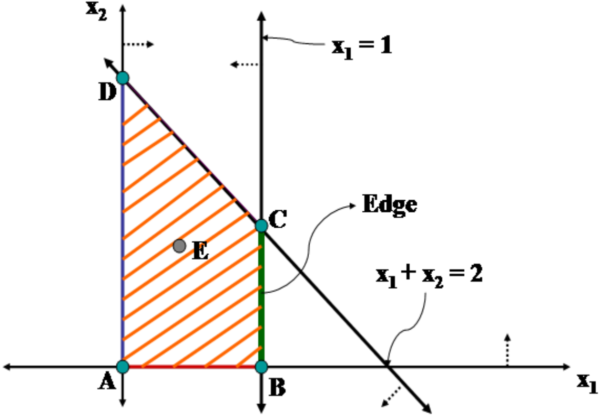
- Constraint Set:
  $X = \{(x_1, x_2) : x_1 + x_2 \leq 2, x_1 \leq 1, x_1 \geq 0, x_2 \geq 0\}$
- 4 constraints in $\mathbb{R}^2$



- Two constraints are *binding* (active) at every extreme point
- Fewer than two constraints are binding at other points

Consider the constraint set: $X = \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$ where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ and $\text{rank}(\boldsymbol{A}) = m$.

- $m + n$ hyperplanes associated with $m + n$ halfspaces
- $m + n$ halfspaces define X
- An extreme point lies on n linearly independent defining hyperplanes of X
- If X is nonempty, the set of extreme points of X is not empty and has a finite number of points.
- An edge of X is formed by intersection of $n - 1$ linearly independent hyperplanes
- Two extreme points of X are said to be adjacent if the line segment joining them is an edge of X



- For example, B and C are *adjacent* points
- Adjacent extreme points have $n-1$ common binding linearly independent hyperplanes

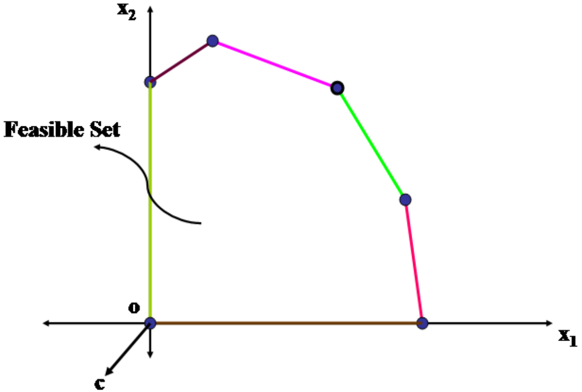
Remarks:
Consider the constraint set: $X = \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$ where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ and $\text{rank}(\boldsymbol{A}) = m$.

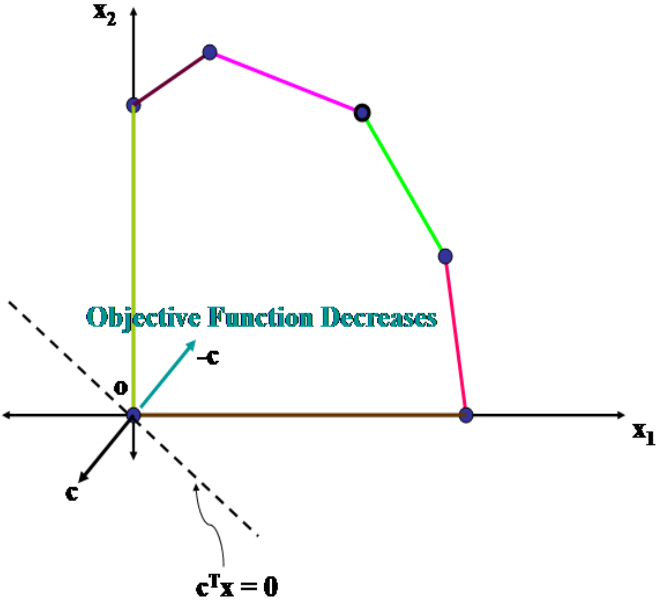
- Let $\bar{\boldsymbol{x}} \in X$ be an extreme point of X
- m equality constraints are active at $\bar{\boldsymbol{x}}$
- Therefore, $n - m$ additional hyperplanes (from the non-negativity constraints) are active at $\bar{\boldsymbol{x}}$

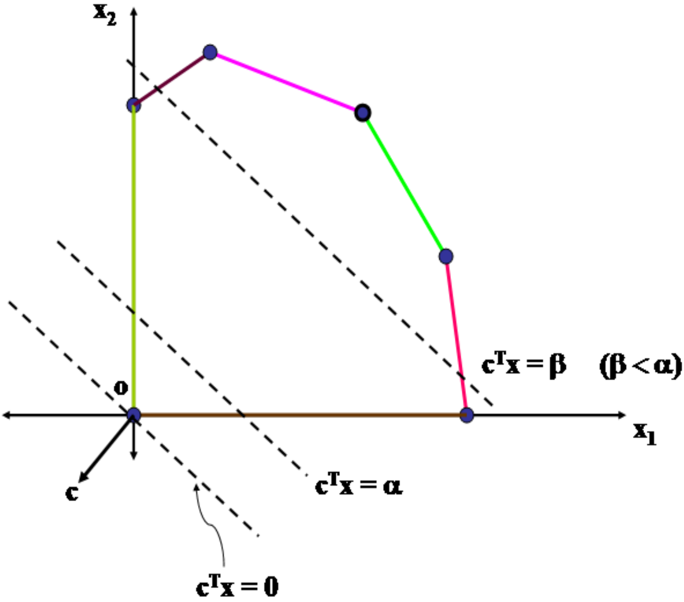
Geometric Solution of a LP:

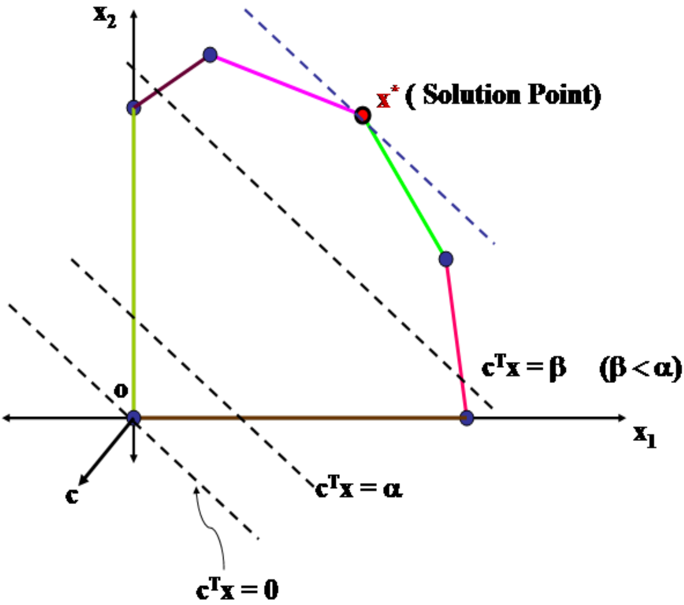
$$\begin{aligned} \min \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

where $\boldsymbol{A} \in \mathbb{R}^{m\times n}, \boldsymbol{c} \in \mathbb{R}^n$ and $\boldsymbol{b} \in \mathbb{R}^m$.

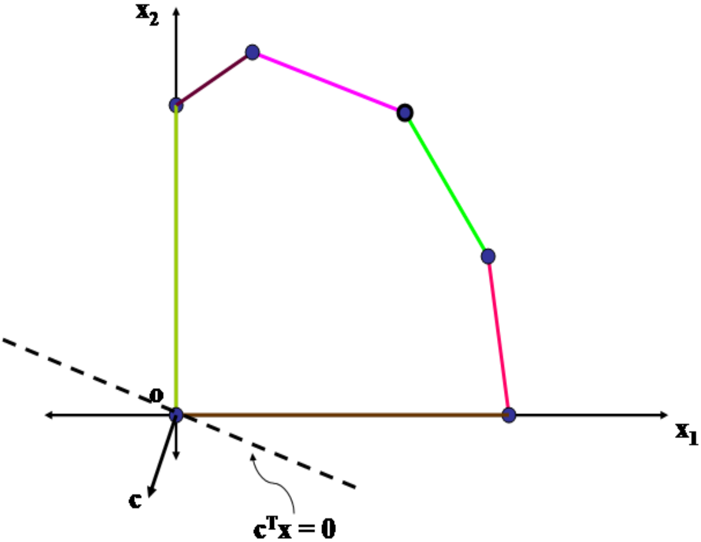


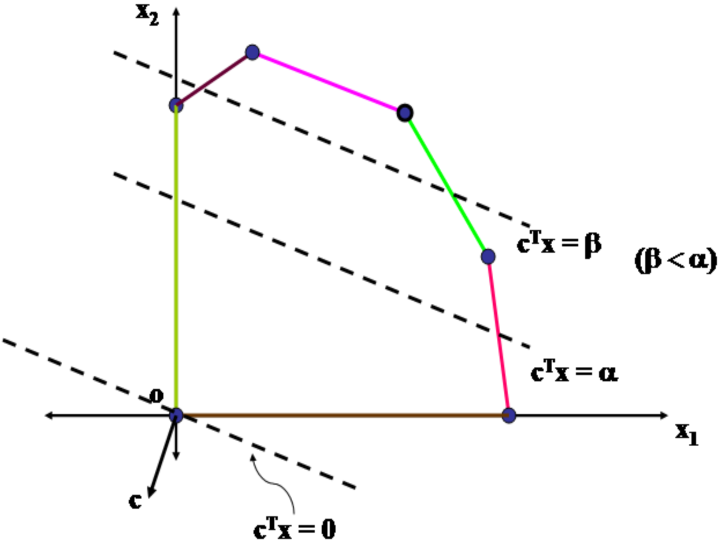

x_2
Objective Function Decreases
$-c$
0
x_1
c
$c^T x = 0$

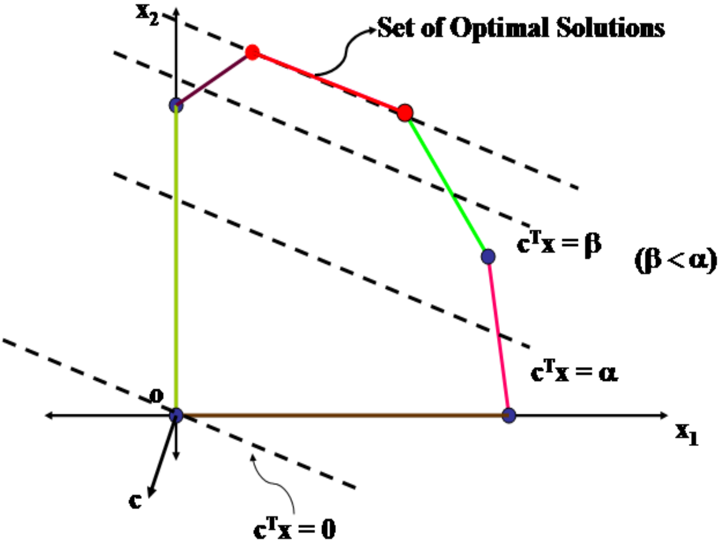

x_2
$c^Tx = \beta \quad (\beta < \alpha)$
0
x_1
c
$c^Tx = \alpha$
$c^Tx = 0$



x_2
x^* (Solution Point)
$c^T x = \beta \quad (\beta < \alpha)$
0
x_1
c
$c^T x = \alpha$
$c^T x = 0$


x_2
0
x_1
c
$c^T x = 0$

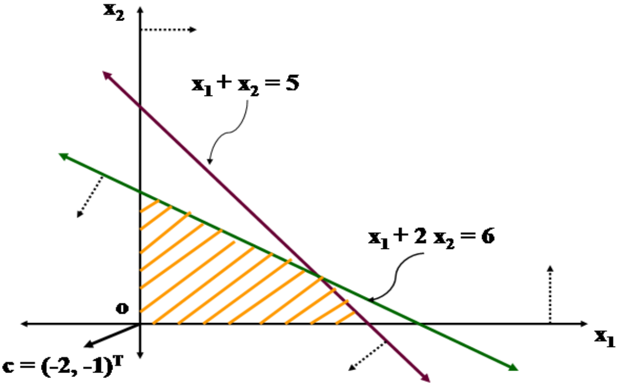

x_2
$c^T x = \beta$
$(\beta < \alpha)$
$c^T x = \alpha$
o
x_1
c
$c^T x = 0$



x_2
Set of Optimal Solutions
$c^T x = \beta$
$(\beta < \alpha)$
$c^T x = \alpha$
o
x_1
c
$c^T x = 0$

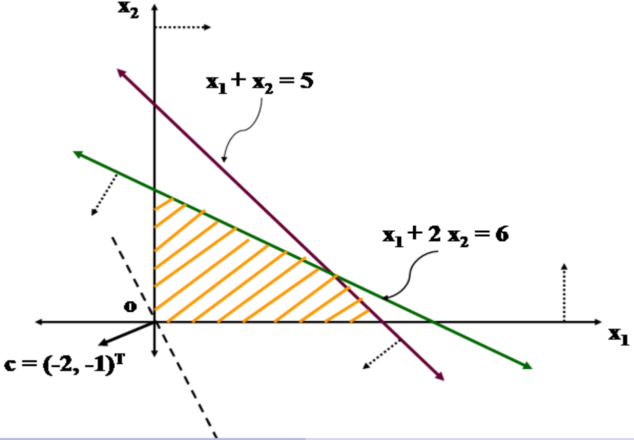
Example:

$$\begin{aligned} \min \quad & -2x_1 - x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 5 \\ & x_1 + 2x_2 \leq 6 \end{aligned}$$



Example:

$$
\begin{aligned}
\min \quad & -2x_1 - x_2 \\
\text{s.t.} \quad & x_1 + x_2 \leq 5 \\
& x_1 + 2x_2 \leq 6 \\
& x_1, x_2 \geq 0
\end{aligned}
$$



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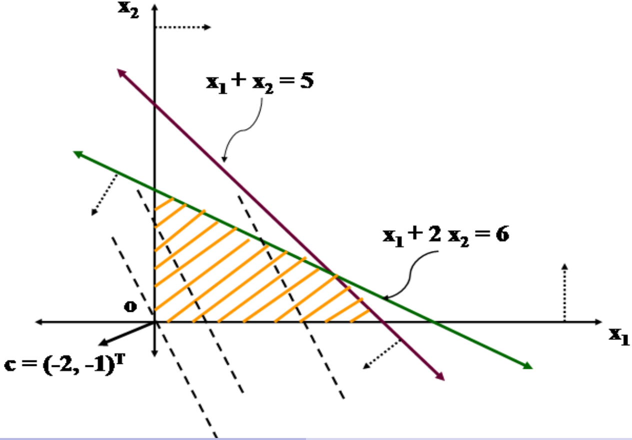

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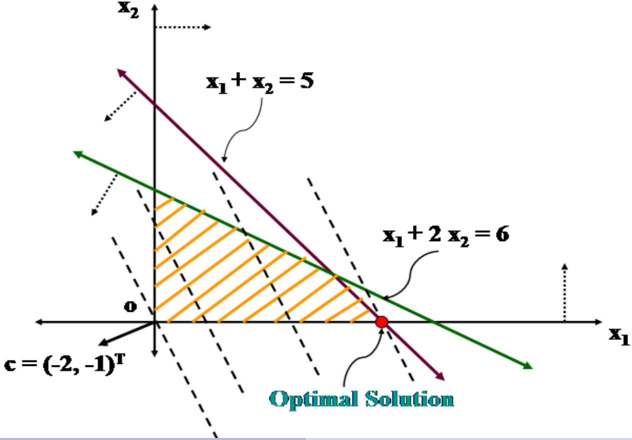
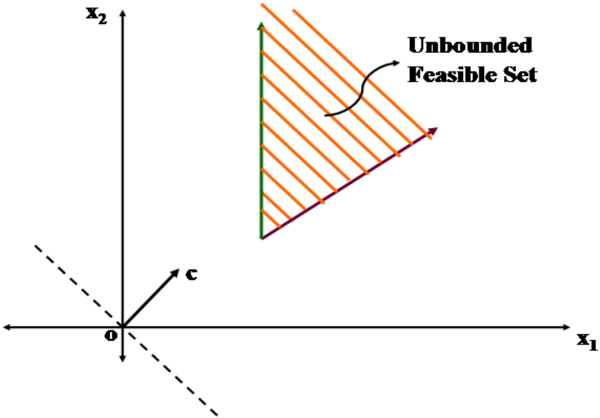


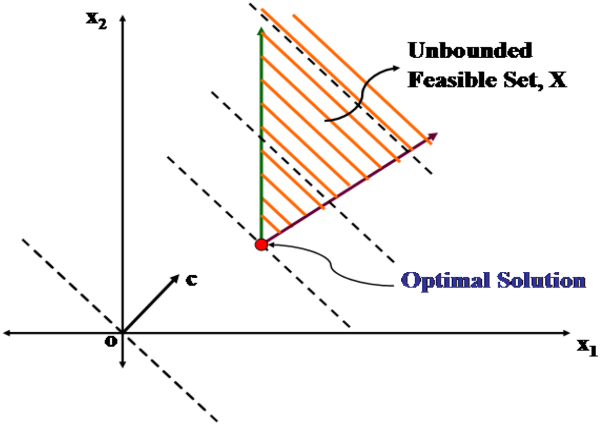
Example:

$$\min \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad \boldsymbol{x} \in X$$



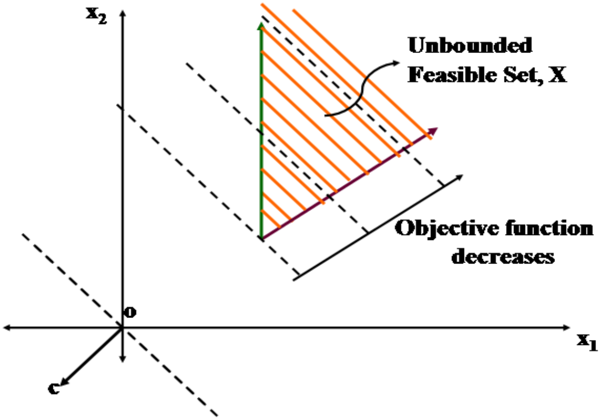
Example:

$$\begin{aligned} \min \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{x} \in X \end{aligned}$$



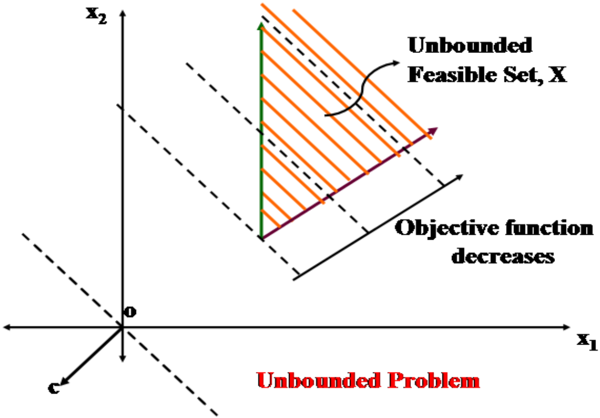
Example:

$$\begin{aligned} \min \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{x} \in X \end{aligned}$$



Example:

$$\min \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad \boldsymbol{x} \in X$$



Consider a linear programming problem **LP**:

$$\begin{aligned} \min \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{a}_i^T \boldsymbol{x} \ (\leq, =, \geq) \ \ \boldsymbol{b}_i, \quad i = 1, \ldots, m \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

Let $X = \{\boldsymbol{x} : \boldsymbol{a}_i^T \boldsymbol{x} \ (\leq, =, \geq) \ \boldsymbol{b}_i, \ \ i = 1, \ldots, m, \boldsymbol{x} \geq \boldsymbol{0}\}$.
Remarks:

- X is a closed convex set
- The set of optimal solutions is a convex set.
- The linear program may have *no solution* or a *unique solution* or *infinitely many solutions*.
- If $\boldsymbol{x}^*$ is an optimal solution to **LP**, then $\boldsymbol{x}^*$ must be a *boundary* point of X. If $z = \boldsymbol{c}^T \boldsymbol{x}^*$, then $\{\boldsymbol{x} : \boldsymbol{c}^T \boldsymbol{x} = z\}$ is a supporting hyperplane to X.
- If X is compact and if there is an optimal solution to **LP**, then *at least one* extreme point of X is an optimal solution to the linear programming problem.

LP in Standard Form:

$$\begin{aligned} \min \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ and $\text{rank}(\boldsymbol{A}) = m$.

Assumption: Feasible set is non-empty

- Any linear program can be converted to the Standard Form.

(a) $\max\ \boldsymbol{c}^T\boldsymbol{x} = -\min\ -\boldsymbol{c}^T\boldsymbol{x}$

(b) Constraint of the type

$$\boldsymbol{a}^T\boldsymbol{x} \leq b,\ \boldsymbol{x} \geq \boldsymbol{0}$$

can be written as

$$\begin{aligned} \boldsymbol{a}^T\boldsymbol{x} + y &= b \\ \boldsymbol{x} &\geq \boldsymbol{0} \\ y &\geq 0 \end{aligned}$$

(c) Constraint of the type

$$\boldsymbol{a}^T\boldsymbol{x} \geq b,\ \boldsymbol{x} \geq \boldsymbol{0}$$

can be written as

$$\begin{aligned} \boldsymbol{a}^T\boldsymbol{x} - z &= b \\ \boldsymbol{x} &\geq \boldsymbol{0} \\ z &\geq 0 \end{aligned}$$

(d) Free variables ($x_i \in \mathbb{R}$) can be defined as

$$x_i = x_i^+ - x_i^-,\ \ x_i^+ \geq 0,\ x_i^- \geq 0$$

Example:

$$
\begin{aligned}
\min \quad & x_1 - 2x_2 - 3x_3 \\
\text{s.t.} \quad & x_1 + 2x_2 + x_3 \leq 14 \\
& x_1 + 2x_2 + 4x_3 \geq 12 \\
& x_1 - x_2 + x_3 = 2 \\
& x_1, x_2 \text{ unrestricted} \\
& x_3 \leq -3
\end{aligned}
$$

- Write the constraints as *equality* constraints
  - $x_1 + 2x_2 + x_3 + x_4 = 14,\ x_4 \geq 0$
  - $x_1 + 2x_2 + 4x_3 - x_5 = 12,\ x_5 \geq 0$
- Define new variables $x_1^+, x_1^-, x_2^+, x_2^-$ and x_3' such that
  - $x_1 = x_1^+ - x_1^-,$ where $x_1^+ \geq 0, x_1^- \geq 0$
  - $x_2 = x_2^+ - x_2^-,$ where $x_2^+ \geq 0, x_2^- \geq 0$
  - $x_3' = -3 - x_3$ so that $x_3' \geq 0$

Therefore, the program

$$\begin{aligned} \min \quad & x_1 - 2x_2 - 3x_3 \\ \text{s.t.} \quad & x_1 + 2x_2 + x_3 \leq 14 \\ & x_1 + 2x_2 + 4x_3 \geq 12 \\ & x_1 - x_2 + x_3 = 2 \\ & x_1, x_2 \text{ unrestricted} \\ & x_3 \leq -3 \end{aligned}$$

can be converted to the standard form:

$$\begin{aligned} \min \quad & x_1^+ - x_1^- - 2(x_2^+ - x_2^-) + 3(3 + x_3') \\ \text{s.t.} \quad & x_1^+ - x_1^- + 2(x_2^+ - x_2^-) - x_3' + x_4 = 17 \\ & x_1^+ - x_1^- + 2(x_2^+ - x_2^-) - 4x_3' - x_5 = 24 \\ & x_1^+ - x_1^- - x_2^+ + x_2^- - x_3' = 5 \\ & x_1^+, x_1^-, x_2^+, x_2^-, x_3', x_4, x_5 \geq 0 \end{aligned}$$

Consider the linear program in standard form (**SLP**):

$$\begin{aligned} \min \quad & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

where $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\operatorname{rank}(\boldsymbol{A}) = \operatorname{rank}(\boldsymbol{A}|\boldsymbol{b}) = m$.
Let $\boldsymbol{B} \in \mathbb{R}^{m\times m}$ be formed using m linearly independent columns of $\boldsymbol{A}$.
Therefore, the system of equations, $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ can be written as,

$$(\boldsymbol{B} \;\; \boldsymbol{N}) \begin{pmatrix} \boldsymbol{x}_B \\ \boldsymbol{x}_N \end{pmatrix} = \boldsymbol{b}.$$

Letting $\boldsymbol{x}_N = \boldsymbol{0}$, we get

$$\boldsymbol{B}\boldsymbol{x}_B = \boldsymbol{b} \;\Rightarrow\; \boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b}. \qquad (\boldsymbol{x}_B : \text{ Basic Variables})$$

$(\boldsymbol{x}_B \;\; \boldsymbol{0})^T$: Basic solution w.r.t. the *basis matrix* $\boldsymbol{B}$

Basic Feasible Solution

If $\boldsymbol{x}_B \geq \mathbf{0}$, then $(\boldsymbol{x}_B \ \ \mathbf{0})^T$ is called a *basic feasible solution* of

$$\begin{aligned} \boldsymbol{A}\boldsymbol{x} &= \boldsymbol{b} \\ \boldsymbol{x} &\geq \mathbf{0} \end{aligned}$$

w.r.t. the basis matrix $\boldsymbol{B}$.

Theorem

Let $X = \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \geq \mathbf{0}\}$. $\boldsymbol{x}$ is an extreme point of X if and only if $\boldsymbol{x}$ is a basic feasible solution of

$$\begin{aligned} \boldsymbol{A}\boldsymbol{x} &= \boldsymbol{b} \\ \boldsymbol{x} &\geq \mathbf{0}. \end{aligned}$$

Proof.

(a) Let $\boldsymbol{x}$ be a basic feasible solution of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}$.

Therefore, $\boldsymbol{x} = (\underbrace{x_1, \ldots, x_m}_{\geq 0}, \underbrace{0, \ldots, 0}_{n-m})$. Let $\boldsymbol{B} = (\boldsymbol{a}_1 | \boldsymbol{a}_2 | \ldots | \boldsymbol{a}_m)$ where $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$ are linearly independent. So,

$$x_1 \boldsymbol{a}_1 + \ldots + x_m \boldsymbol{a}_m = \boldsymbol{b}.$$

Suppose $\boldsymbol{x}$ is not an extreme point of X.
Let $\boldsymbol{y}, \boldsymbol{z} \in X$, $\boldsymbol{y} \neq \boldsymbol{z}$ and $\boldsymbol{x} = \alpha \boldsymbol{y} + (1 - \alpha)\boldsymbol{z}$, $0 < \alpha < 1$.
Since $\boldsymbol{y}, \boldsymbol{z} \geq \boldsymbol{0}$, we have

$$\left.\begin{array}{l} y_{m+1} = \ldots = y_n = 0 \\ z_{m+1} = \ldots = z_n = 0 \end{array}\right\} \text{ and } \begin{array}{l} y_1 \boldsymbol{a}_1 + \ldots + y_m \boldsymbol{a}_m = \boldsymbol{b} \\ z_1 \boldsymbol{a}_1 + \ldots + z_m \boldsymbol{a}_m = \boldsymbol{b} \end{array}$$

Since $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$ are linearly independent, $\boldsymbol{x} = \boldsymbol{y} = \boldsymbol{z}$, a contradiction. So, $\boldsymbol{x}$ is an extreme point of X.

Proof.(continued)

(b) Let $\boldsymbol{x}$ be an extreme point of X.

$\boldsymbol{x} \in X \;\Rightarrow\; \boldsymbol{Ax} = \boldsymbol{b},\; \boldsymbol{x} \geq \boldsymbol{0}$.

There exist n linearly independent constraints active at $\boldsymbol{x}$.

- m active constraints associated with $\boldsymbol{Ax} = \boldsymbol{b}$.
- $n - m$ active constraints associated with $n - m$ non-negativity constraints

$\boldsymbol{x}$ is the *unique* solution of $\boldsymbol{Ax} = \boldsymbol{b},\; \boldsymbol{x}_N = \boldsymbol{0}$.

$$\boldsymbol{Ax} = \boldsymbol{b} \;\Rightarrow\; \boldsymbol{Bx}_B + \boldsymbol{Nx}_N = \boldsymbol{b} \;\Rightarrow\; \boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b} \geq \boldsymbol{0}$$

Therefore, $\boldsymbol{x} = (\boldsymbol{x}_B \;\; \boldsymbol{x}_N)^T$ is a basic feasible solution. □

$$\text{Number of basic solutions} \;\leq \binom{n}{m}$$

Enough to search the finite set of vertices of X to get an optimal

Theorem

Let X be non-empty and compact constraint set of a linear program. Then, an optimal solution to the linear program exists and it is attained at a vertex of X.

Proof.

Objective function, $\boldsymbol{c}^T\boldsymbol{x}$, of the linear program is continuous and the constraint set is compact. Therefore, by Weierstrass' Theorem, optimal solution exists.

The set of vertices, $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k\}$, of X is finite.

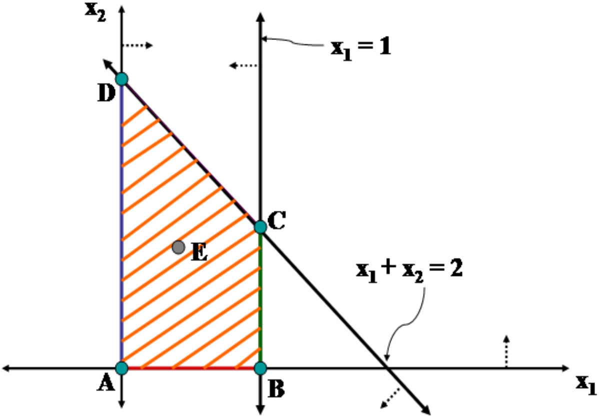
Therefore, X is the convex hull of $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$.

Hence, for every $\boldsymbol{x} \in X, \boldsymbol{x} = \sum_{i=1}^{k} \alpha_i \boldsymbol{x}_i$ where $\alpha_i \geq 0, \sum_{i=1}^{k} \alpha_i = 1$.

Let $z^* = \min_{1 \leq i \leq k} \ \boldsymbol{c}^T\boldsymbol{x}_i$. Therefore, for any $\boldsymbol{x} \in X$, $z = \boldsymbol{c}^T\boldsymbol{x} = \sum_{i=1}^{k} \alpha_i \boldsymbol{c}^T\boldsymbol{x}_i \geq z^* \sum_{i=1}^{k} \alpha_i = z^*$. So, the minimum value of $\boldsymbol{c}^T\boldsymbol{x}$ is attained at a vertex of X. $\square$

Consider the constraints:

$$x_1 + x_2 \le 2$$
$$x_1 \le 1$$
$$x_1, x_2 \ge 0$$



The given constraints

$$\begin{aligned} x_1 + x_2 &\leq 2 \\ x_1 &\leq 1 \\ x_1, x_2 &\geq 0 \end{aligned}$$

can be written in the form, $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \ \boldsymbol{x} \geq \boldsymbol{0}$:

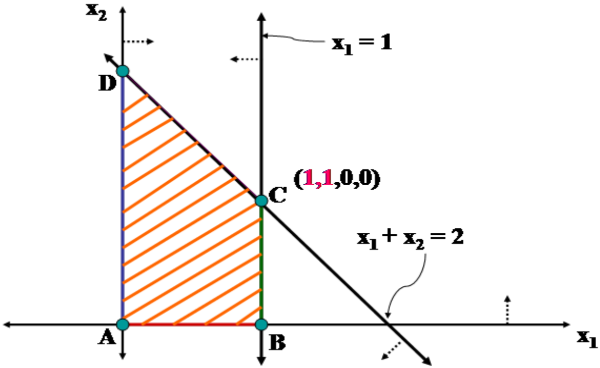
$$\begin{array}{lllll} x_1 & +x_2 + x_3 & & & = 2 \\ x_1 & & & +x_4 & = 1 \\ x_1, & x_2, & x_3, & x_4 & \geq 0 \end{array}$$

Let $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = (\boldsymbol{a}_1 | \boldsymbol{a}_2 | \boldsymbol{a}_3 | \boldsymbol{a}_4)$ and $\boldsymbol{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = (\boldsymbol{a}_1|\boldsymbol{a}_2|\boldsymbol{a}_3|\boldsymbol{a}_4) \text{ and } \boldsymbol{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

(1) $\boldsymbol{B} = (\boldsymbol{a}_1|\boldsymbol{a}_2) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

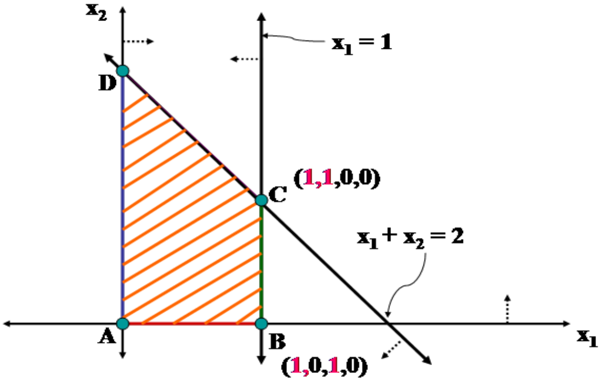
$\boldsymbol{x}_B = (x_1\ x_2)^T = \boldsymbol{B}^{-1}\boldsymbol{b} = (1\ 1)^T$ and $\boldsymbol{x}_N = (x_3\ x_4)^T = (0\ 0)^T$.
$\boldsymbol{x} = (\boldsymbol{x}_B\ \boldsymbol{x}_N)^T$ corresponds to the vertex **C**.



$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = (\boldsymbol{a}_1|\boldsymbol{a}_2|\boldsymbol{a}_3|\boldsymbol{a}_4) \text{ and } \boldsymbol{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

(2) $\boldsymbol{B} = (\boldsymbol{a}_1|\boldsymbol{a}_3) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

$\boldsymbol{x}_B = (x_1\ x_3)^T = \boldsymbol{B}^{-1}\boldsymbol{b} = (1\ 1)^T$ and $\boldsymbol{x}_N = (x_2\ x_4)^T = (0\ 0)^T$.
$\boldsymbol{x} = (\boldsymbol{x}_B\ \boldsymbol{x}_N)^T$ corresponds to the vertex **B**.



$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = (\boldsymbol{a}_1 | \boldsymbol{a}_2 | \boldsymbol{a}_3 | \boldsymbol{a}_4) \text{ and } \boldsymbol{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

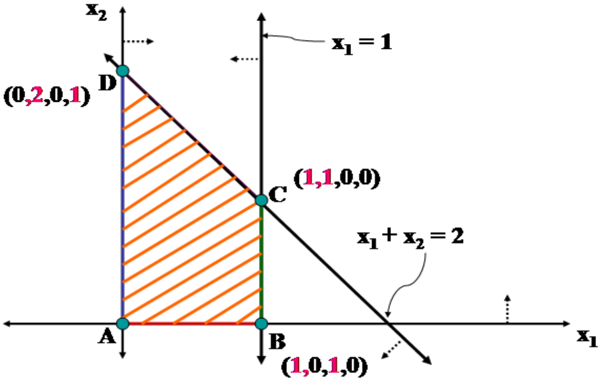
$$(3) \ \boldsymbol{B} = (\boldsymbol{a}_1 | \boldsymbol{a}_4) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$\boldsymbol{x}_B = (x_1 \ x_4)^T = \boldsymbol{B}^{-1}\boldsymbol{b} = (2 \ -1)^T$ and $\boldsymbol{x}_N = (x_2 \ x_3)^T = (0 \ 0)^T$.
$\boldsymbol{x} = (\boldsymbol{x}_B \ \boldsymbol{x}_N)^T$ is not a basic feasible point

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = (\boldsymbol{a}_1|\boldsymbol{a}_2|\boldsymbol{a}_3|\boldsymbol{a}_4) \text{ and } \boldsymbol{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

(4) $\boldsymbol{B} = (\boldsymbol{a}_2|\boldsymbol{a}_4) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

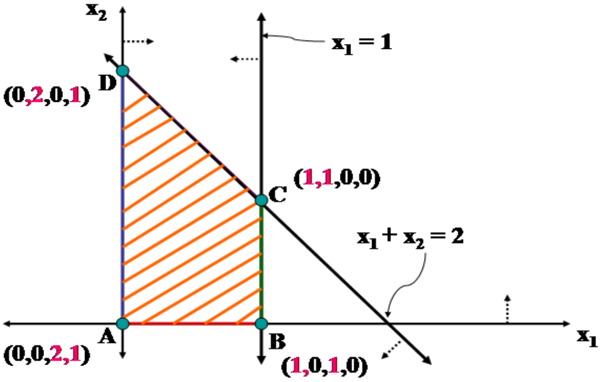
$\boldsymbol{x}_B = (x_2\ x_4)^T = \boldsymbol{B}^{-1}\boldsymbol{b} = (2\ 1)^T$ and $\boldsymbol{x}_N = (x_1\ x_3)^T = (0\ 0)^T$.
$\boldsymbol{x} = (\boldsymbol{x}_B\ \boldsymbol{x}_N)^T$ corresponds to the vertex **D**.



$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = (\boldsymbol{a}_1 | \boldsymbol{a}_2 | \boldsymbol{a}_3 | \boldsymbol{a}_4) \text{ and } \boldsymbol{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

(5) $\boldsymbol{B} = (\boldsymbol{a}_3 | \boldsymbol{a}_4) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\boldsymbol{x}_B = (x_3\ x_4)^T = \boldsymbol{B}^{-1}\boldsymbol{b} = (2\ 1)^T$ and $\boldsymbol{x}_N = (x_1\ x_2)^T = (0\ 0)^T$.
$\boldsymbol{x} = (\boldsymbol{x}_B\ \boldsymbol{x}_N)^T$ corresponds to the vertex **A**.



Example:

$$\begin{aligned} \min \quad & -3x_1 - x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 2 \\ & x_1 \leq 1 \end{aligned}$$