Numerical Optimization

Linear Programming - II

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NPTEL Course on Numerical Optimization

LP in Standard Form:

$$
\begin{aligned}
\min \quad & \boldsymbol{c}^T\boldsymbol{x} \\
\text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\
& \boldsymbol{x} \geq \boldsymbol{0}
\end{aligned}
$$

where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ and $\text{rank}(\boldsymbol{A}) = m$.

Let $\boldsymbol{x}$ be a *nondegenerate basic feasible solution* corresponding to the basic variable set B and non-basic variable set N.
Let $\boldsymbol{B}$ denote the basis matrix.

$$\begin{aligned} \boldsymbol{A}\boldsymbol{x} &= \boldsymbol{b} \\ \therefore \boldsymbol{B}\boldsymbol{x}_B + \boldsymbol{N}\boldsymbol{x}_N &= \boldsymbol{b} \\ \therefore \boldsymbol{x}_B &= \boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}\boldsymbol{N}\boldsymbol{x}_N \end{aligned}$$

Particular Solution: $\boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b}$ and $\boldsymbol{x}_N = \boldsymbol{0}$

$$\begin{aligned} \text{Objective Function} &= \boldsymbol{c}^T\boldsymbol{x} \\ &= \boldsymbol{c}_B^T\boldsymbol{x}_B + \boldsymbol{c}_N^T\boldsymbol{x}_N \\ &= \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{N}\boldsymbol{x}_N + \boldsymbol{c}_N^T\boldsymbol{x}_N \\ &= \bar{z} + \bar{\boldsymbol{c}}_B^T\boldsymbol{x}_B + \bar{\boldsymbol{c}}_N^T\boldsymbol{x}_N \end{aligned}$$

where $\bar{\boldsymbol{c}}_B^T = \boldsymbol{0}^T$ and $\bar{\boldsymbol{c}}_N^T = \boldsymbol{c}_N^T - \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{N}$ are the *relative cost factors* corresponding to the basis matrix $\boldsymbol{B}$ and $\bar{z}$ denotes the current objective function value.

LP in Standard Form:

$$\begin{aligned} \min \quad & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ and $\text{rank}(\boldsymbol{A}) = m$.

- Convex Programming Problem
- Assumption:
  - Feasible set is non-empty
  - Slater's condition is satisfied
- First order KKT conditions are necessary and sufficient at optimality

LP in Standard Form:

$$\begin{aligned} \min \quad & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ and $\text{rank}(\boldsymbol{A}) = m$.

Define the Lagrangian function:

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = \boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{\mu}^T(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}) - \boldsymbol{\lambda}^T\boldsymbol{x}$$

First Order KKT Conditions at optimality:

- Primal Feasibility: $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}$
- $\nabla_x \mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = \boldsymbol{0} \;\Rightarrow\; \boldsymbol{A}^T\boldsymbol{\mu} + \boldsymbol{\lambda} = \boldsymbol{c}$
- Complementary Slackness Condition: $\lambda_i x_i = 0 \; \forall \; i$
- Non-negativity: $\lambda_i \geq 0 \; \forall \; i$

Let $\boldsymbol{x}$ be a *nondegenerate basic feasible solution* corresponding to the basic variable set B and non-basic variable set N.
$\boldsymbol{x} = (\boldsymbol{x}_B \;\; \boldsymbol{x}_N)^T$ where $\boldsymbol{x}_B > \boldsymbol{0}$ and $\boldsymbol{x}_N = \boldsymbol{0}$.
At optimal $(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu})$,

- $\boldsymbol{\lambda}_B = \boldsymbol{0}$ and $\boldsymbol{\lambda}_N \geq \boldsymbol{0}$.
- $\boldsymbol{c} = \boldsymbol{A}^T \boldsymbol{\mu} + \boldsymbol{\lambda}$. That is,

$$\begin{pmatrix} \boldsymbol{c}_B \\ \boldsymbol{c}_N \end{pmatrix} = \begin{pmatrix} \boldsymbol{B}^T \\ \boldsymbol{N}^T \end{pmatrix} \boldsymbol{\mu} + \begin{pmatrix} \boldsymbol{\lambda}_B \\ \boldsymbol{\lambda}_N \end{pmatrix} \Rightarrow \quad \begin{array}{c} \boldsymbol{c}_B = \boldsymbol{B}^T \boldsymbol{\mu} + \boldsymbol{\lambda}_B \\ \boldsymbol{c}_N = \boldsymbol{N}^T \boldsymbol{\mu} + \boldsymbol{\lambda}_N \end{array}$$

- $\boldsymbol{\lambda}_B = \boldsymbol{0} \;\Rightarrow\; \boldsymbol{c}_B = \boldsymbol{B}^T \boldsymbol{\mu} \;\Rightarrow\; \boldsymbol{\mu} = \boldsymbol{B}^{T^{-1}} \boldsymbol{c}_B$
- $\boldsymbol{\lambda}_N \geq \boldsymbol{0}$ requires that

$$\boldsymbol{\lambda}_N = \boldsymbol{c}_N - (\boldsymbol{B}^{-1} \boldsymbol{N})^T \boldsymbol{c}_B \geq \boldsymbol{0}$$

The current basic feasible solution $\boldsymbol{x}$ is *not* optimal if there exists $x_q \in N$ such that $\lambda_q < 0$.

$\boldsymbol{x}$ is feasible $\Rightarrow$ $\boldsymbol{Ax} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}$.

$\therefore \boldsymbol{Bx}_B + \boldsymbol{Nx}_N = \boldsymbol{b} \Rightarrow \boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}\boldsymbol{Nx}_N$

$$
\begin{aligned}
\text{Objective Function at } \boldsymbol{x} &= \boldsymbol{c}^T\boldsymbol{x} \\
&= \boldsymbol{c}_B^T\boldsymbol{x}_B + \boldsymbol{c}_N^T\boldsymbol{x}_N \\
&= \boldsymbol{c}_B^T(\boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}\boldsymbol{Nx}_N) + \boldsymbol{c}_N^T\boldsymbol{x}_N \\
&= \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{b} + \underbrace{(\boldsymbol{c}_N^T - \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{N})}_{\boldsymbol{\lambda}_N^T}\boldsymbol{x}_N \\
&= \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{b} \qquad (\because \boldsymbol{x}_N = \boldsymbol{0})
\end{aligned}
$$

Suppose there exists a non-basic variable $x_q \in N$ ($x_q = 0$) such that $\lambda_q < 0$.

$$\therefore \text{Objective Function at } \boldsymbol{x} = \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{b} + \lambda_q x_q$$

The objective function can be decreased if x_q is changed from 0 to some positive value (by making x_q a *basic* variable).

Suppose x_q is made a basic variable. Therefore, some existing basic variable $x_p (\in B)$ needs to be made *non-basic*.

$$\begin{aligned} \boldsymbol{Ax} &= \boldsymbol{b} \\ \therefore \boldsymbol{Bx}_B + \boldsymbol{Nx}_N &= \boldsymbol{b} \\ \therefore \boldsymbol{x}_B &= \boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}\boldsymbol{Nx}_N \\ \therefore \begin{pmatrix} x_1 \\ \vdots \\ x_p \\ \vdots \\ x_m \end{pmatrix} &= \begin{pmatrix} \bar{b}_1 \\ \vdots \\ \bar{b}_p \\ \vdots \\ \bar{b}_m \end{pmatrix} - (\boldsymbol{B}^{-1}\boldsymbol{N})_{\cdot,q} x_q \end{aligned}$$

Note: $\bar{\boldsymbol{b}} > \boldsymbol{0}$
How to choose x_p?
Require $(\boldsymbol{B}^{-1}\boldsymbol{N})_{p,q} > 0$ so that x_p can decrease if x_q increases.

$$\begin{pmatrix} x_1 \\ \vdots \\ x_p \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} \bar{b}_1 \\ \vdots \\ \bar{b}_p \\ \vdots \\ \bar{b}_m \end{pmatrix} - (\boldsymbol{B}^{-1}\boldsymbol{N})_{\cdot,q} x_q$$

How to choose the basic variable x_p to be made non-basic?
Let $\hat{B}_q = \{x_j \in B : (\boldsymbol{B}^{-1}\boldsymbol{N})_{jq} > 0\}$.

$$x_q = \min_{x_j \in \hat{B}_q} \frac{\bar{b}_j}{(\boldsymbol{B}^{-1}\boldsymbol{N})_{jq}} = \frac{\bar{b}_p}{(\boldsymbol{B}^{-1}\boldsymbol{N})_{pq}}$$

- If $\hat{B}_q = \phi$, the problem is *unbounded*.

Summary of steps:

Given a basic feasible solution, $(\boldsymbol{x}_B\ \boldsymbol{x}_N)^T$ w.r.t. $\boldsymbol{B}$.

If $\boldsymbol{\lambda}_N = \boldsymbol{c}_N - (\boldsymbol{B}^{-1}\boldsymbol{N})^T\boldsymbol{c}_B \leq \boldsymbol{0}$

1. Choose $x_q \in N$ such that $\lambda_q < 0$
2. $\hat{B}_q = \{x_j \in B : (\boldsymbol{B}^{-1}\boldsymbol{N})_{jq} > 0\}$.
3. If $\hat{B}_q \neq \phi$, find the basic variable x_p to be made non-basic.

$$x_q = \min_{x_j \in \hat{B}_q} \frac{\bar{b}_j}{(\boldsymbol{B}^{-1}\boldsymbol{N})_{jq}} = \frac{\bar{b}_p}{(\boldsymbol{B}^{-1}\boldsymbol{N})_{pq}}$$

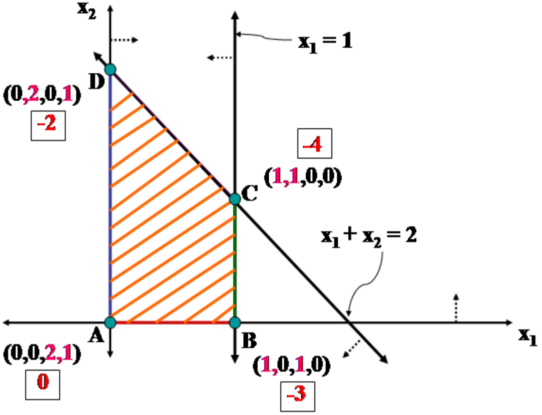
4. $x_i = \bar{b}_i - (\boldsymbol{B}^{-1}\boldsymbol{N})_{iq}x_q,\ \forall\, x_i \in B$
5. Swap x_p and x_q from the sets B and N

Geometrical Interpretation of Steps 1-5

Moving from one extreme point of the feasible set to an adjacent extreme point so that the objective function value decreases.

Example:

$$\begin{aligned}\min \quad & -3x_1 - x_2 \\ \text{s.t.} \quad & x_1 + x_2 \le 2 \\ & x_1 \le 1 \\ & x_1, x_2 \ge 0\end{aligned}$$



LP in Standard Form:

$$\begin{aligned} \min \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ and $\text{rank}(\boldsymbol{A}) = m$.

Given a basis matrix $\boldsymbol{B}$ w.r.t. the basis vector set B,

- Basic Feasible Solution: $\boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b}, \ \boldsymbol{x}_N = \boldsymbol{0}$
- Objective function = $\boldsymbol{c}_B^T \boldsymbol{B}^{-1}\boldsymbol{b}$
- Relative cost factors, $\boldsymbol{\lambda}_N = \boldsymbol{c}_N - (\boldsymbol{B}^{-1}N)^T \boldsymbol{c}_B$

LP in Standard Form:

$$
\begin{aligned}
\min \quad & \boldsymbol{c}^T\boldsymbol{x} \\
\text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\
& \boldsymbol{x} \geq \boldsymbol{0}
\end{aligned}
$$

where $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ and $\text{rank}(\boldsymbol{A}) = m$.

$$
\begin{array}{c}
\\ x_B \\ \\
\end{array}
\left(\begin{array}{c|c|c}
\text{Basic Variables} & \text{Nonbasic Variables} & \text{RHS} \\
\hline\hline
\boldsymbol{B} & \boldsymbol{N} & \boldsymbol{b} \\
\hline
\boldsymbol{c}_B^T & \boldsymbol{c}_N^T & 0 \\
\hline
\end{array}\right)
$$

$$
\left(\begin{array}{c|c|c}
\boldsymbol{I} & \boldsymbol{B}^{-1}\boldsymbol{N} & \boldsymbol{B}^{-1}\boldsymbol{b} \\
\hline
\boldsymbol{c}_B^T & \boldsymbol{c}_N^T & 0 \\
\hline
\end{array}\right)
$$

$$
\left(\begin{array}{c|c|c}
\boldsymbol{I} & \boldsymbol{B}^{-1}\boldsymbol{N} & \boldsymbol{B}^{-1}\boldsymbol{b} \\
\hline
\boldsymbol{0}^T & \boldsymbol{c}_N^T - \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{N} & \text{-}\ \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{b} \\
\hline
\end{array}\right)
$$

Example:

$$\begin{aligned} \min \quad & -3x_1 - x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 2 \\ & x_1 \leq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Given problem in the standard form:

$$\begin{aligned} \min \quad & -3x_1 - x_2 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 = 2 \\ & x_1 + x_4 = 1 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

- Initial Basic Feasible Solution:
  $\boldsymbol{x}_B = (x_3, x_4)^T = (2, 1)^T$, $\boldsymbol{x}_N = (x_1, x_2)^T = (0, 0)^T$

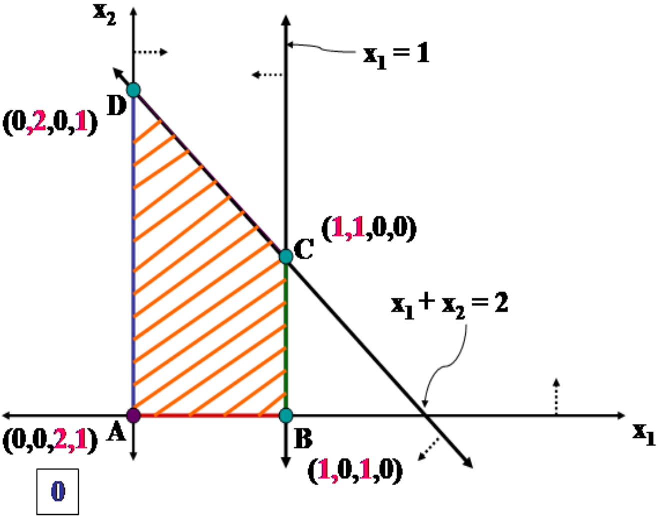
$$\begin{aligned} \min \quad & -3x_1 - x_2 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 = 2 \\ & x_1 + x_4 = 1 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

- Initial Basic Feasible Solution:
  $\boldsymbol{x}_B = (x_3, x_4)^T = (2, 1)^T,\ \boldsymbol{x}_N = (x_1, x_2)^T = (0, 0)^T$

Initial Tableau:

$$\left(\begin{array}{cccc|c} x_1 & x_2 & \color{red}{x_3} & \color{red}{x_4} & \text{RHS} \\ \hline\hline 1 & 1 & 1 & 0 & 2 \\ 1 & 0 & 0 & 1 & 1 \\ \hline -3 & -1 & 0 & 0 & 0 \end{array}\right)$$

- Initial Basic Feasible Solution:
  $\boldsymbol{x}_B = (x_3, x_4)^T = (2, 1)^T,\ \boldsymbol{x}_N = (x_1, x_2)^T = (0, 0)^T$
- Current Objective function = 0



$$\left(\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \text{RHS} \\ \hline\hline 1 & 1 & 1 & 0 & 2 \\ 1 & 0 & 0 & 1 & 1 \\ \hline -3 & -1 & 0 & 0 & 0 \end{array}\right)$$

Incoming basic variable: x_1

$$\left(\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \text{RHS} \\ \hline\hline 1 & 1 & 1 & 0 & 2 \\ 1 & 0 & 0 & 1 & 1 \\ \hline -3 & -1 & 0 & 0 & 0 \end{array}\right)$$

Incoming nonbasic variable: x_1

$$\left(\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \text{RHS} \\ \hline\hline 1 & 1 & 1 & 0 & 2 \\ 1 & 0 & 0 & 1 & 1 \\ \hline -3 & -1 & 0 & 0 & 0 \end{array}\right)$$

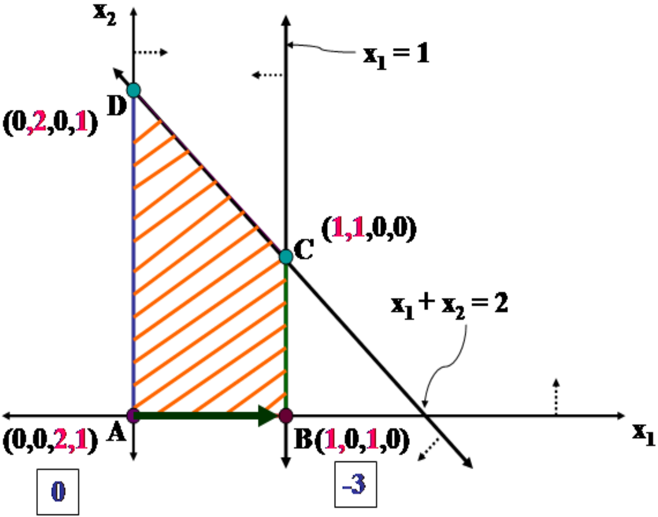
Outgoing basic variable: x_4

$$\left(\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \text{RHS} \\ \hline\hline 0 & 1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ \hline 0 & -1 & 0 & 3 & 3 \end{array}\right)$$

$$\left(\begin{array}{cccc|c}
x_1 & x_2 & x_3 & x_4 & \text{RHS} \\
\hline\hline
0 & 1 & 1 & -1 & 1 \\
1 & 0 & 0 & 1 & 1 \\
\hline
0 & -1 & 0 & 3 & 3
\end{array}\right)$$

- Current Basic Feasible Solution: $(x_1, x_3)^T = (1, 1)^T,\ (x_2, x_4)^T = (0, 0)^T$
- Current Objective function = -3

- Current Basic Feasible Solution: $(x_1, x_3)^T = (1, 1)^T, \ (x_2, x_4)^T = (0, 0)^T$
- Current Objective function = -3



$$\left(\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \text{RHS} \\ \hline\hline 0 & 1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ \hline 0 & -1 & 0 & 3 & 3 \end{array}\right)$$

Incoming nonbasic variable: x_2

$$\left(\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \text{RHS} \\ \hline\hline 0 & 1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ \hline 0 & -1 & 0 & 3 & 3 \end{array}\right)$$

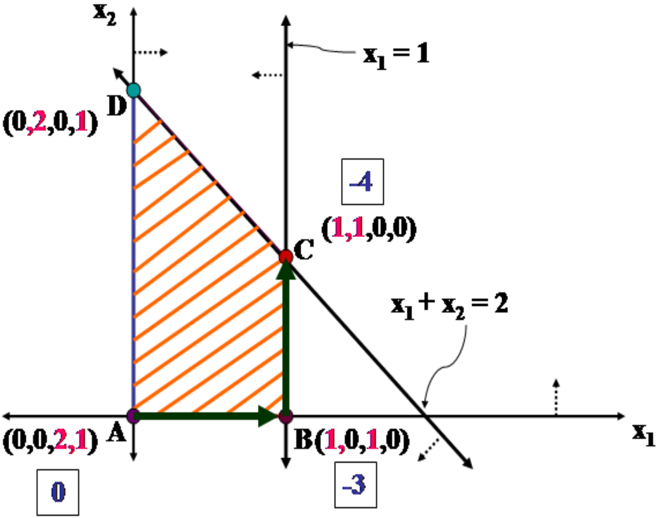
Incoming nonbasic variable: x_2

$$\left(\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \text{RHS} \\ \hline\hline 0 & 1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ \hline 0 & -1 & 0 & 3 & 3 \end{array}\right)$$

Outgoing basic variable: x_3

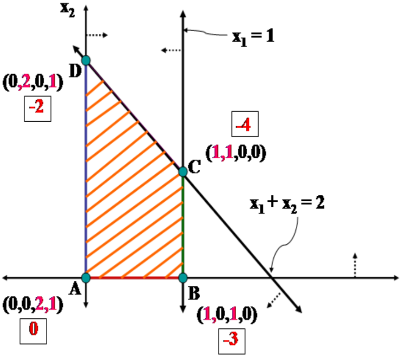
$$\left(\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \text{RHS} \\ \hline\hline 0 & 1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 1 & 2 & 4 \end{array}\right)$$

- Current Basic Feasible Solution:
  $(x_1, x_2)^T = (1, 1)^T, \ (x_3, x_4)^T = (0, 0)^T$
- Current Objective function = -4

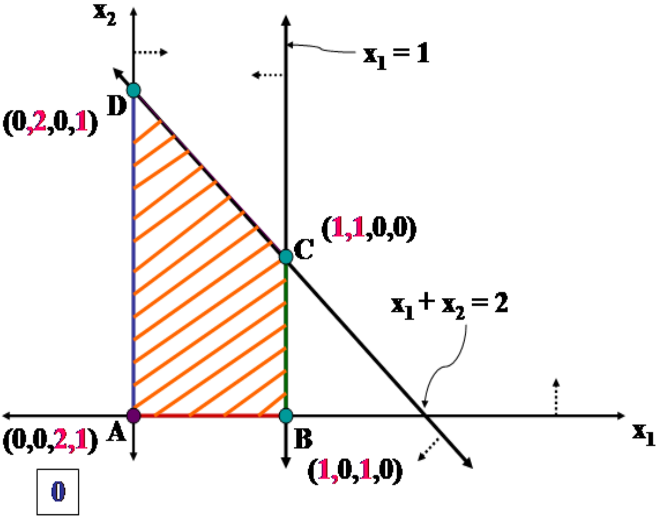


$$\left(\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \text{RHS} \\ \hline\hline 0 & 1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 1 & 2 & 4 \end{array}\right)$$

- Current Basic Feasible Solution: $(x_1, x_2)^T = (1,1)^T,\ (x_3, x_4)^T = (0,0)^T$
- Current Objective function = -4
- $\boldsymbol{\lambda}_N = (1,2)^T > \mathbf{0}^T$
- Optimal Solution: $\boldsymbol{x}^* = (1,1)^T, \boldsymbol{c}^T\boldsymbol{x}^* = -4$

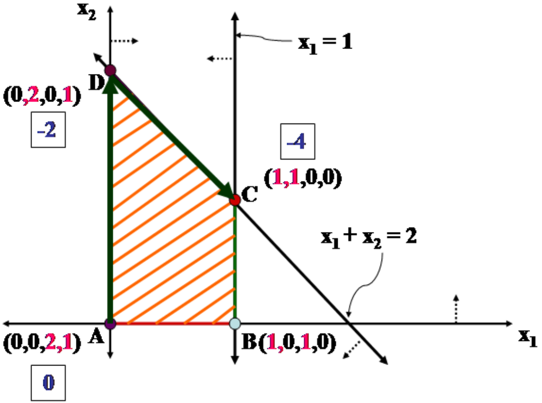


- Initial Basic Feasible Solution:
  $\boldsymbol{x}_B = (x_3, x_4)^T = (2, 1)^T, \; \boldsymbol{x}_N = (x_1, x_2)^T = (0, 0)^T$
- Current Objective function = 0



$$\left(\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \text{RHS} \\ \hline 1 & 1 & 1 & 0 & 2 \\ 1 & 0 & 0 & 1 & 1 \\ \hline -3 & -1 & 0 & 0 & 0 \end{array}\right)$$

Incoming basic variable: x_2



LP in Standard Form:

$$\begin{aligned} \min \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ and $\text{rank}(\boldsymbol{A}) = m$.

Simplex Algorithm to solve an LP [Dantzig][1]

[1] G. B. Dantzig, *Linear Programming and Extensions*, Princeton University Press, 1963

Simplex Algorithm (to solve an LP in Standard Form)

(1) Get initial basis matrix $\boldsymbol{B}$, basic feasible solution, $\boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b}$, $\boldsymbol{x}_N = \boldsymbol{0}$, set *Unbounded* = FALSE.

(2) $\boldsymbol{\lambda}_N = \boldsymbol{c}_N - (\boldsymbol{B}^{-1}\boldsymbol{N})^T \boldsymbol{c}_B$

(3) **while** ($Unbounded$ == FALSE and $\exists\, x_j \in N \ni \lambda_j < 0$)

  (a) Select a non-basic variable $x_q \in N$ such that $\lambda_q < 0$

  (b) $\hat{B}_q = \{x_j \in B : (\boldsymbol{B}^{-1}\boldsymbol{N})_{jq} > 0\}$.

  (c) **if** $\hat{B}_q == \phi$

   - *Unbounded* = TRUE

  else

   (i) $x_q = \min_{x_j \in \hat{B}_q} \dfrac{\bar{b}_j}{(\boldsymbol{B}^{-1}\boldsymbol{N})_{jq}} = \dfrac{\bar{b}_p}{(\boldsymbol{B}^{-1}\boldsymbol{N})_{pq}}$

   (ii) $x_i = \bar{b}_i - (\boldsymbol{B}^{-1}\boldsymbol{N})_{iq} x_q, \ \forall\, x_i \in B$

   (iii) Swap x_p and x_q between B and N, update $\boldsymbol{B}$ and $\boldsymbol{\lambda}$

  endif

endwhile

Output : $\boldsymbol{x}^* = (\boldsymbol{x}_B, \boldsymbol{x}_N)^T$ if *Unbounded* == FALSE

Remarks:

- How to select a non-basic variable $x_q \in N$ in Step 3(a) of the Simplex Algorithm?
$$x_q = \operatorname*{argmin}_{x_j \in N : \lambda_j < 0} \lambda_j$$
- If the basic feasible solution is *nondegenerate* at each iteration, then the Simplex Algorithm terminates in a finite number of iterations.
$$\begin{aligned} \text{Objective Function} &= \boldsymbol{c}_B^T \boldsymbol{x}_B + \boldsymbol{c}_N^T \boldsymbol{x}_N \\ &= \boldsymbol{c}_B^T \boldsymbol{B}^{-1} \boldsymbol{b} - \boldsymbol{c}_B^T \boldsymbol{B}^{-1} \boldsymbol{N} \boldsymbol{x}_N + \boldsymbol{c}_N^T \boldsymbol{x}_N \\ &= \bar{z} + \bar{\boldsymbol{c}}_B^T \boldsymbol{x}_B + \bar{\boldsymbol{c}}_N^T \boldsymbol{x}_N \end{aligned}$$
where $\bar{\boldsymbol{c}}_B^T = \boldsymbol{0}^T$, $\bar{\boldsymbol{c}}_N^T = \boldsymbol{c}_N^T - \boldsymbol{c}_B^T \boldsymbol{B}^{-1} \boldsymbol{N}$ and $\bar{z}$ denotes the current objective function value. If the basic feasible solution is nondegenerate, then the objective function decreases in each iteration. Since the number of basic feasible solutions is finite, the algorithm has finite convergence.

- How to get initial basic feasible solution?

(a) Case I: Given constraints

$$\begin{aligned} \boldsymbol{A}\boldsymbol{x} &\leq \boldsymbol{b} \\ \boldsymbol{x} &\geq \boldsymbol{0} \end{aligned}$$

(with $\boldsymbol{b} \geq \boldsymbol{0}$), can be written as

$$\begin{aligned} \boldsymbol{A}\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{b} \\ \boldsymbol{x}, \boldsymbol{y} \geq \boldsymbol{0} \end{aligned} \quad \Rightarrow \quad \begin{aligned} (\boldsymbol{A} \;\; \boldsymbol{I}) \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} = \boldsymbol{b} \\ \boldsymbol{x}, \boldsymbol{y} \geq \boldsymbol{0}. \end{aligned}$$

Use Basic feasible solution, $(\boldsymbol{x}, \boldsymbol{y})^T = (\boldsymbol{0}, \boldsymbol{b})^T$, to solve

$$\begin{aligned} \min \quad & \boldsymbol{c}^T\boldsymbol{x} + \boldsymbol{0}^T\boldsymbol{y} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{b} \\ & \boldsymbol{x}, \boldsymbol{y} \geq \boldsymbol{0} \end{aligned}$$

(b) Case 2: Given constraints (**C2**)

$$\begin{aligned} \boldsymbol{A}\boldsymbol{x} &= \boldsymbol{b} \\ \boldsymbol{x} &\geq \boldsymbol{0}, \end{aligned}$$

(with $\boldsymbol{b} \geq \boldsymbol{0}$),
solve the *artificial* linear program (**ALP**)

$$\begin{aligned} \min \quad & \boldsymbol{1}^T \boldsymbol{y} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{b} \\ & \boldsymbol{x}, \boldsymbol{y} \geq \boldsymbol{0}. \end{aligned}$$

to get the initial basic feasible solution, $\boldsymbol{x} \geq \boldsymbol{0}$ (if it exists).

- If there exists $\boldsymbol{x}$ that satisfies constraints **C2**, then **ALP** has the optimal objective function value of 0 with $\boldsymbol{y} = \boldsymbol{0}$ and $\boldsymbol{x} \geq \boldsymbol{0}$.
- If **C2** has no feasible solution, then the optimal objective function value of **ALP** is *greater than* 0.

(c) Case 3: Given constraints

$$\begin{aligned} \boldsymbol{Ax} &\geq \boldsymbol{b} \\ \boldsymbol{x} &\geq \boldsymbol{0} \end{aligned}$$

(with $\boldsymbol{b} \geq \boldsymbol{0}$), can be written as

$$\begin{array}{c} \boldsymbol{Ax} - \boldsymbol{z} + \boldsymbol{y} = \boldsymbol{b} \\ \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \geq \boldsymbol{0} \end{array} \quad \Rightarrow \quad \begin{array}{c} (\boldsymbol{A} \;\; \boldsymbol{I} \;\; -\boldsymbol{I}) \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \end{pmatrix} = \boldsymbol{b} \\ \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \geq \boldsymbol{0}. \end{array}$$

- Solve an artificial linear program (similar to Case 2)

Two Phase Method

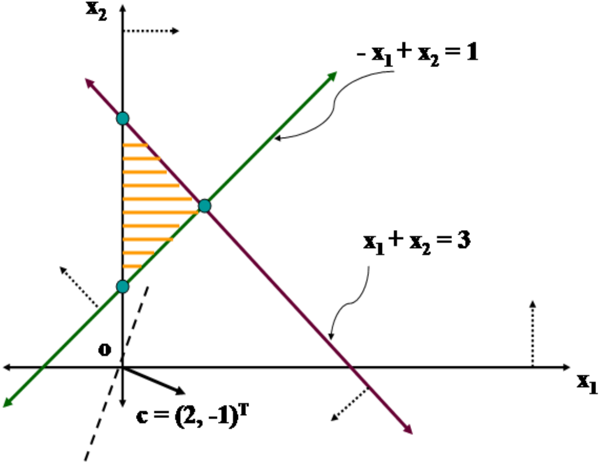
Given the linear program (**SLP**):

$$\begin{aligned} \min \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

where $\boldsymbol{b} \geq \boldsymbol{0}$.

- Phase I: Introduce artificial variables and solve the artificial linear program with initial basic feasible solution, $(\boldsymbol{x} = \boldsymbol{0}, \boldsymbol{y} = \boldsymbol{b})$, to get a basic feasible solution for **SLP** or conclude that it does not exist.
- Phase II: Get the initial basic feasible solution (if it exists) from Phase I to solve **SLP**.

Example:
$$\begin{aligned} \min \quad & 2x_1 - x_2 \\ \text{s.t.} \quad & x_1 + x_2 \le 3 \\ & -x_1 + x_2 \ge 1 \\ & x_1, x_2 \ge 0 \end{aligned}$$



Example:

$$\begin{aligned} \min \quad & 2x_1 - x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 3 \\ & -x_1 + x_2 \geq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Phase I: Introducing artificial variables, the constraints become

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ -x_1 + x_2 - x_4 + x_5 &= 1 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

Therefore, the artificial linear program

$$\begin{aligned} \min \quad & x_3 + x_5 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 = 3 \\ & -x_1 + x_2 - x_4 + x_5 = 1 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

has $x_3 = 3, x_5 = 1, x_1 = x_2 = x_4 = 0$ as its initial basic feasible solution.

$$\begin{array}{ll} \min & x_3 + x_5 \\ \text{s.t.} & x_1 + x_2 + x_3 = 3 \\ & -x_1 + x_2 - x_4 + x_5 = 1 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{array}$$

Initial Tableau:

$$\left(\begin{array}{ccccc|c} x_1 & x_2 & \color{red}{x_3} & x_4 & \color{red}{x_5} & \text{RHS} \\ \hline\hline 1 & 1 & 1 & 0 & 0 & 3 \\ -1 & 1 & 0 & -1 & 1 & 1 \\ \hline 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right)$$

Making the relative costs of basic variables 0,

$$\left(\begin{array}{ccccc|c} x_1 & x_2 & \color{red}{x_3} & x_4 & \color{red}{x_5} & \text{RHS} \\ \hline\hline 1 & 1 & 1 & 0 & 0 & 3 \\ -1 & 1 & 0 & -1 & 1 & 1 \\ \hline 0 & -2 & 0 & 1 & 0 & -4 \end{array}\right)$$

$$\left(\begin{array}{ccccc|c}
x_1 & x_2 & x_3 & x_4 & x_5 & \text{RHS} \\
\hline\hline
1 & 1 & 1 & 0 & 0 & 3 \\
-1 & 1 & 0 & -1 & 1 & 1 \\
\hline
0 & -2 & 0 & 1 & 0 & -4
\end{array}\right)$$

Incoming non-basic variable: x_2

$$\left(\begin{array}{ccccc|c}
x_1 & x_2 & x_3 & x_4 & x_5 & \text{RHS} \\
\hline\hline
1 & 1 & 1 & 0 & 0 & 3 \\
-1 & 1 & 0 & -1 & 1 & 1 \\
\hline
0 & -2 & 0 & 1 & 0 & -4
\end{array}\right)$$

Outgoing basic variable: x_5

$$\left(\begin{array}{ccccc|c}
x_1 & x_2 & x_3 & x_4 & x_5 & \text{RHS} \\
\hline\hline
2 & 0 & 1 & 1 & -1 & 2 \\
-1 & 1 & 0 & -1 & 1 & 1 \\
\hline
-2 & 0 & 0 & -1 & 2 & -2
\end{array}\right)$$

$$\left(\begin{array}{ccccc|c}
x_1 & x_2 & x_3 & x_4 & x_5 & \text{RHS} \\
\hline\hline
2 & 0 & 1 & 1 & -1 & 2 \\
-1 & 1 & 0 & -1 & 1 & 1 \\
\hline
-2 & 0 & 0 & -1 & 2 & -2
\end{array}\right)$$

Incoming non-basic variable: x_1

$$\left(\begin{array}{ccccc|c}
x_1 & x_2 & x_3 & x_4 & x_5 & \text{RHS} \\
\hline\hline
2 & 0 & 1 & 1 & -1 & 2 \\
-1 & 1 & 0 & -1 & 1 & 1 \\
\hline
-2 & 0 & 0 & -1 & 2 & -2
\end{array}\right)$$

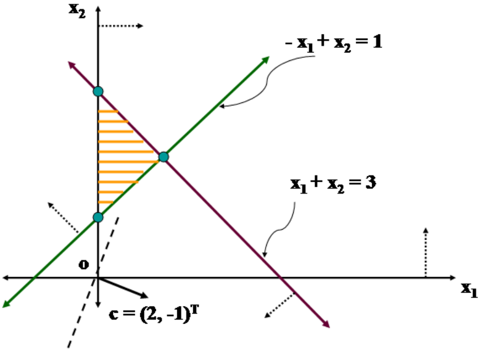
Outgoing basic variable: x_3

$$\left(\begin{array}{ccccc|c}
x_1 & x_2 & x_3 & x_4 & x_5 & \text{RHS} \\
\hline\hline
1 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 1 \\
0 & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 2 \\
\hline
0 & 0 & 1 & 0 & 1 & 0
\end{array}\right)$$

Basic feasible solution: $x_1 = 1, x_2 = 2, x_3 = x_4 = 0.$

$$\left(\begin{array}{ccccc|c} \color{red}{x_1} & \color{red}{x_2} & x_3 & x_4 & x_5 & \text{RHS} \\ \hline\hline 1 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 1 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 2 \\ \hline 0 & 0 & 1 & 0 & 1 & \color{magenta}{0} \end{array}\right)$$

Basic feasible solution: $x_1 = 1, x_2 = 2, x_3 = x_4 = 0$.



Phase II: For the given problem,

$$\begin{aligned} \min \quad & 2x_1 - x_2 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 = 3 \\ & -x_1 + x_2 - x_4 = 1 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

Initial Tableau:

$$\left(\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \text{RHS} \\ \hline\hline 1 & 0 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 2 \\ \hline 2 & -1 & 0 & 0 & 0 \end{array}\right)$$

Making the relative costs of basis variables 0,

$$\left(\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \text{RHS} \\ \hline\hline 1 & 0 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 2 \\ \hline 0 & 0 & -\frac{1}{2} & -\frac{3}{2} & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \text{RHS} \\ \hline\hline 1 & 0 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 2 \\ \hline 0 & 0 & -\frac{1}{2} & -\frac{3}{2} & 0 \end{array}\right)$$

Incoming non-basic variable: x_4

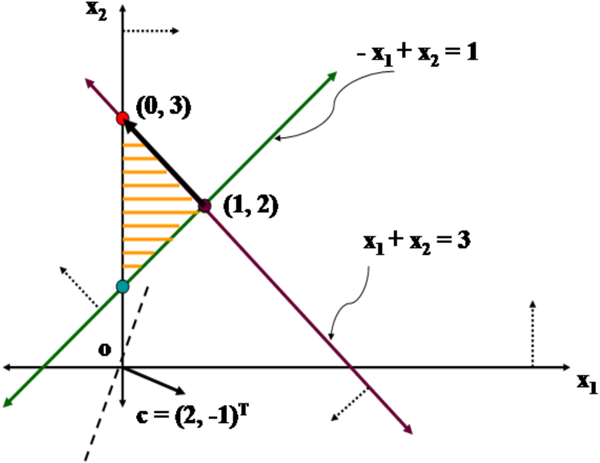
$$\left(\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \text{RHS} \\ \hline\hline 1 & 0 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 2 \\ \hline 0 & 0 & -\frac{1}{2} & -\frac{3}{2} & 0 \end{array}\right)$$

Outgoing basic variable: x_1

$$\left(\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \text{RHS} \\ \hline\hline 2 & 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 0 & 3 \\ \hline 3 & 0 & 1 & 0 & 3 \end{array}\right)$$

$\boldsymbol{x}^* = (x_1, x_2)^T = (0, 3)^T,\ \boldsymbol{c}^T\boldsymbol{x}^* = -3$

$$\left(\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \text{RHS} \\ \hline\hline 2 & 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 0 & 3 \\ \hline 3 & 0 & 1 & 0 & 3 \end{array}\right)$$



Example:

$$\begin{aligned} \min \quad & -4x_1 - 2x_2 \\ \text{s.t.} \quad & 3x_1 - 2x_2 \geq 4 \\ & -2x_1 + x_2 = 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Phase I: Introducing artificial variables, the constraints become

$$\begin{aligned} 3x_1 - 2x_2 - x_3 + x_4 &= 4 \\ -2x_1 + x_2 + x_5 &= 2 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

Therefore, the artificial linear program

$$\begin{aligned} \min \quad & x_4 + x_5 \\ \text{s.t.} \quad & 3x_1 - 2x_2 - x_3 + x_4 = 4 \\ & -2x_1 + x_2 + x_5 = 2 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

has $x_4 = 4, x_5 = 2, x_1 = x_2 = x_3 = 0$ as its initial basic feasible solution.

$$\begin{aligned} \min \quad & x_4 + x_5 \\ \text{s.t.} \quad & 3x_1 - 2x_2 - x_3 + x_4 = 4 \\ & -2x_1 + x_2 + x_5 = 2 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

Initial Tableau:

x_1	x_2	x_3	x_4	x_5	RHS
3	-2	-1	1	0	4
-2	1	0	0	1	2
0	0	0	1	1	0

Making the relative costs of basic variables 0,

x_1	x_2	x_3	x_4	x_5	RHS
3	-2	-1	1	0	4
-2	1	0	0	1	2
-1	1	1	0	0	-6

$$\left(\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \text{RHS} \\ \hline\hline 3 & -2 & -1 & 1 & 0 & 4 \\ -2 & 1 & 0 & 0 & 1 & 2 \\ \hline -1 & 1 & 1 & 0 & 0 & -6 \end{array}\right)$$

Incoming non-basic variable: x_1

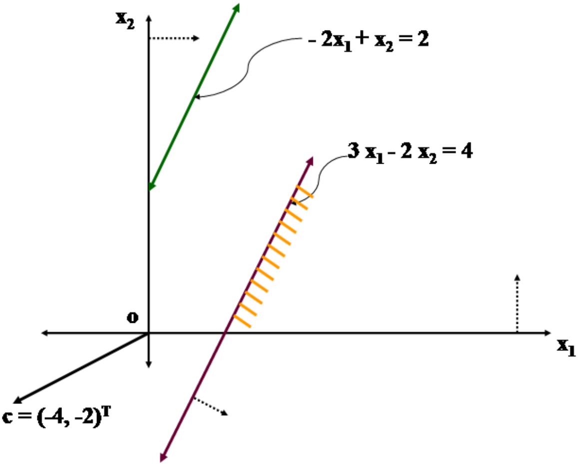
$$\left(\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \text{RHS} \\ \hline\hline 3 & -2 & -1 & 1 & 0 & 4 \\ -2 & 1 & 0 & 0 & 1 & 2 \\ \hline -1 & 1 & 1 & 0 & 0 & -6 \end{array}\right)$$

Outgoing basic variable: x_4

$$\left(\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \text{RHS} \\ \hline\hline 1 & \frac{-2}{3} & \frac{-1}{3} & \frac{1}{3} & 0 & \frac{4}{3} \\ 0 & \frac{-1}{3} & \frac{-2}{3} & \frac{2}{3} & 1 & \frac{14}{3} \\ \hline 0 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & -\frac{14}{3} \end{array}\right)$$

The given problem has no feasible solution.

$$
\begin{aligned}
\min \quad & -4x_1 - 2x_2 \\
\text{s.t.} \quad & 3x_1 - 2x_2 \geq 4 \\
& -2x_1 + x_2 = 2 \\
& x_1, x_2 \geq 0
\end{aligned}
$$



- Cycling
  - Possible in the presence of degenerate basic feasible solution
  - Consider the example given by Beale[2]

$$
\begin{aligned}
\min \quad & -\tfrac{3}{4}x_1 + 20x_2 - \tfrac{1}{2}x_3 + 6x_4 \\
\text{s.t.} \quad & \tfrac{1}{4}x_1 - 8x_2 - x_3 + 9x_4 = 0 \\
& \tfrac{1}{2}x_1 - 12x_2 - \tfrac{1}{2}x_3 + 3x_4 = 0 \\
& x_3 + x_5 = 1 \\
& x_j \geq 0 \ \forall\, j
\end{aligned}
$$

  - Bland's rule to avoid cycling: Arrange the variables in some ordered sequence. Among all the candidate variables to enter (or to leave) the basis, choose the variable with the smallest index

[2] Beale E.M.L., Cycling in the dual simplex algorithm, *Naval Research Logistics Quarterly*, 2(4), pp. 269-276, 1955.