NPTEL lectures on

Elementary Numerical Analysis

by

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Examination 1

Time: 2 hours | Marks: 30

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1. Let $f : [a, b] \to \mathbb{R}$ and $x_0, x_1, \cdots, x_k$ be $k+1$ distinct points in $[a, b]$. Show that there exists a unique polynomial p_k of degree $\leq k$ such that

$$p_k(x_j) = f(x_j), \; j = 0, 1, \cdots, k.$$

(3 marks)

2. Define the divided difference $f[x_0, x_1, \cdots, x_k]$ as the coefficient of x^k in p_k in Q.1. Prove the following recurrence formula:

$$f[x_0, x_1, \cdots, x_k] = \frac{f[x_1, x_2, \cdots, x_k] - f[x_0, x_1, \cdots, x_{k-1}]}{x_k - x_0}.$$

(3 marks)

3. Derive the Trapezoidal rule along with the error formula for approximation of $\int_a^b f(x)dx$.

(3 marks)

4. Consider the quadrature rule

$$\int_0^1 f(x)dx \approx \frac{1}{8}\left(f(0) + 3f(\frac{1}{3}) + 3f(\frac{2}{3}) + f(1) \right).$$

Determine the degree of precision of this rule, that is, find the highest degree of polynomial for which the above rule is exact. (3 marks)

5. Let $f : [a, b] \to \mathbb{R}$ be a 4 times continuously differentiable function. For n even, consider

$$a = t_0 < t_1 < \cdots < t_n = b,$$

an uniform partition of $[a, b]$ with

$$h = t_{i+1} - t_i = \frac{b-a}{n}, \; i = 0, 1, \cdots, n-1.$$

Let T_n denote the composite Trapezoidal rule associated with the above partition which approximates $\int_a^b f(x)dx$. Consider the first step of Romberg integration based on T_n and $T_{\frac{n}{2}}$ so as to eliminate the term containing h^2 in the asymptotic expansion. Interprete the result which you obtain as an appropriate numerical quadrature rule. (3 marks)

6. Let $f : [a, b] \to \mathbb{R}$ be a thrice differentiable function and $x_0 \in [a, b]$. Show that

$$\frac{d}{dx} f[x_0,\ x] = f[x_0,\ x,\ x].$$

(3 marks)

7. Let $A = [a_{ij}]$ be an $n \times n$ invertible tridiagonal matrix, that is $a_{ij} = 0$ if $|i - j| > 1$. Compute the number of operations needed to solve the system $Ax = b$ by Gauss elimination without partial pivoting. (3 marks)

8. Find the LU decomposition of the following matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 8 & 11 \\ 3 & 8 & 14 & 20 \\ 4 & 11 & 20 & 30 \end{bmatrix}.$$

(3 marks)

9. Let A be an $n \times n$ invertible matrix such that $A = LU$, where L is a unit lower triangular matrix and U is an upper triangular matrix. For $1 \leq k \leq n$, let A_k denote the principal leading submatrix of A formed by the first k rows and the first k columns of A. Show that $\det(A_k) \neq 0,\ k = 1, 2, \cdots, n$. (3 marks)

10. Let M be an invertible $n \times n$ matrix and $A = MM^T$. Show that A is positive-definite.

(3 marks)