NPTEL lectures on

Elementary Numerical Analysis

by

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Examination 2

Time: 3 hours	Marks: 50

1. Show that if f is k times continuously differentiable on $[a, b]$ and $x_0, x_1, \cdots, x_k$ are distinct points in $[a, b]$, then
$$f[x_0, x_1, \cdots, x_k] = \frac{f^{(k)}(c)}{k!},$$
for some $c \in [a, b]$. (4 marks)

2. Consider
$$\int_{-1}^{1} f(x)dx \approx w_0 f(-1) + w_1 f(x_1) + w_2 f(1).$$
Determine w_0, w_1, w_2 and x_1 such that the formula is exact for cubic polynomials. (4 marks)

3. Show that if a nonsingular linear system $Ax = b$ is altered by multiplication of its j th column by $c \neq 0$, then the solution is altered only in the j th component, which is multiplied by $1/c$.

(2 marks)

4. Let $A = [a_{ij}]$ be an $n \times n$ matrix. Define $\|x\|_1 = \sum_{j=1}^{n} |x_j|$ and $\|A\|_1$ be the induced matrix norm.Show that
$$\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|.$$
(4 marks)

5. Let f be a thrice differentiable function in a neighbourhood of a point a. Show that
$$\left| f'(a) - \frac{f(a+h) - f(a-h)}{2h} \right| \le Ch^2,$$
where C is a constant independent of h. (3 marks)

6. Let A be a 4×4 matrix with eigenvalues $\frac{1}{3}, \frac{1}{2}, 1, 5$. Based on this information find the best possible lower bound for condition number of A with respect to 1 norm. Justify your answer. (3 marks)

7. Let $A = [a_{ij}]$ be an $n \times n$ matrix such that
$$\sum_{j \neq i} |a_{ij}| < |a_{ii}|, \; i = 1, 2, \cdots, n.$$
Define Jacobi iteration method for an iterative solution of $Ax = b$. Let $e^{(k)}$ denote the error in the $k-$th iterate. Show that $\|e^{(k)}\|_\infty \to 0$ as $k \to \infty$. (4 marks)

8. Consider
$$A = \begin{bmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 2 \\ 3 \end{bmatrix}.$$
Choosing the initial approximation to be a zero vector, find the first iterate in Gauss-Seidel method. (3 marks)

9. Let $A = [a_{ij}]$ be a 20×20 tridiagonal matrix such that $a_{ii} = 4, \; i = 1, 2, \cdots, 20$ and $a_{i,i+1} = 1, \; i = 1, 2, \cdots, 19$ and $a_{i-1,i} = 1, \; i = 2, 3, \cdots, 20$. Show that the eigenvalues of A are contained in $[2, 6]$. (3 marks)

10. Define Euler Method and the Midpoint method for approximate solution of the intial value problem
$$y' = f(x, y), \; y(a) = y_0, \;\; x \in [a, b].$$
Obtain the local discretization error in both the methods. (6 marks)

11. Let $g : [a, b] \to [a, b]$ be continuously differentiable and $M = \max_{x \in [a,b]} |g'(x)| < 1$. Let x^* be the unique fixed point of g in $[a, b]$. Let $x_{n+1} = g(x_n), \; x_0 \in [a, b]$. Show that
$$|x_{n+1} - x^*| \leq \frac{M}{1 - M} |x_{n+1} - x_n|.$$
(4 marks)

12. Let A be an $n \times n$ positive definite matrix with distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ arranged in the descending order. Let $Au_j = \lambda_j u_j, \; \|u_j\|_2 = 1, \; j = 1, 2, \cdots, n$. Let x be a non-zero vector such that $\langle x, \, u_1 \rangle \neq 0$. Show that $\frac{A^j x}{\|A^j x\|_2}$ converges to an eigenvector of A associated with λ_1.
(4 marks)

13. Let Q be an $n \times n$ real matrix such that $Q^t Q = I$. Show that the columns of Q are orthonormal.
(3 marks)

14. Let A be an $n \times n$ real matrix such that $A = QR$, where $Q^t Q = I$ and R is an upper triangular matrix. Define $A_1 = RQ$. Show that A and A_1 have the same eigenvalues. (3 marks)
