Note Title 2/28/2012

Elementary Numerical Analysis

Analysis and Implementation of Numerical Algorithms for finding an approximate solution of a Mathematical Problem (using a computer).

Sources of error

1) Uncertainty in physical data
2) Rounding error
3) Discretization error

Rounding error: $\beta = 2, 10, 16$

$$x = \pm(.d_1 d_2 \cdots d_n d_{n+1} \cdots)\beta^e, \quad d_1 \neq 0$$

Floating point representation

$$fl(x) = \begin{cases} \pm(.d_1 d_2 \cdots d_n)\beta^e, & \text{if } 0 \leq d_{n+1} < \frac{\beta}{2} \\ \pm(.d_1 d_2 \cdots (d_n+1))\beta^e, & \text{if } \frac{\beta}{2} \leq d_{n+1} < \beta \end{cases}$$

$$|x - fl(x)| \leq \frac{1}{2}\beta^{e-n}, \quad \frac{|x - fl(x)|}{|x|} \leq \frac{1}{2}\beta^{-n+1}$$

$$fl(x) = x(1+\delta), \quad |\delta| \leq \frac{1}{2}\beta^{-n+1}$$

Chopping error

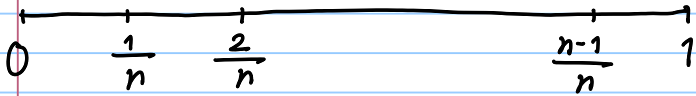
$$x = \pm (.d_1 d_2 \cdots d_n d_{n+1} \cdots)\beta^e, \quad d_1 \neq 0$$

$$fl(x) = \pm (.d_1 d_2 \cdots d_n)\beta^e$$

$$|x - fl(x)| \leq \beta^{e-n}, \quad \frac{|x - fl(x)|}{|x|} \leq \beta^{1-n}$$

$$fl(x) = x(1+\delta), \quad -\beta^{1-n} < \delta \leq 0$$

$f : [0, 1] \longrightarrow \mathbb{R}$ continuous



$$\int_0^1 f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f\left(\frac{i}{n}\right) \frac{1}{n}$$

$$\simeq \sum_{i=1}^{N} f\left(\frac{i}{n}\right) \frac{1}{n}, \quad N : \text{big}$$

<u>Discretization error</u>

Numerical Algorithm:

A complete set of procedures which gives an approximate solution to a mathematical problem.

Algorithm is chosen

Decide the accuracy needed

Estimate the magnitude of the round-off and discretization errors

Number of iterates

Checks on the accuracy

What to do if there is no convergence

Criteria for a 'good' method:

1) Number of computations: Addition / Subtraction, Multiplication / Division
2) Applicable to a class of problems
3) Speed of convergence
4) error management
5) Stability.

Topics :

1) Polynomial / Piecewise polynomial interpolation
2) Numerical Integration and differentiation
3) Solution of a system of linear equations and of eigenvalue problems
4) Finding a root : $f(c) = 0$
5) Initial and Boundary value problems

Reference Books

1. Elementary Numerical Analysis, An Algorithmic Approach by S. D. Conte and Carl de Boor, McGraw Hill International Editions, 1981

2. An Introduction to Numerical Analysis

by

K. E. Atkinson

John Wiley & Sons, New York,

2nd Edition, 1989

Mathematical Preliminaries

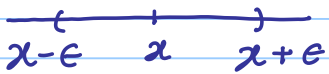
Convergence of a sequence:

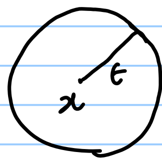
For $n = 1, 2, \cdots$, $x_n \in \mathbb{R} / \mathbb{C}$, $x \in \mathbb{R} / \mathbb{C}$

$x_n \to x$ if for every $\epsilon > 0$ $\exists$ $N \in \mathbb{N}$

such that

$$n \geq N \implies |x_n - x| < \epsilon$$





Continuous functions : Properties

$f : [a, b] \longrightarrow \mathbb{R}$ continuous

$\Rightarrow$ f is bounded

Range of $f = [m, M]$

m : absolute minimum, M : absolute maximum

Intermediate Value Property

$m \leq \alpha \leq M \Rightarrow$ There exists $c \in [a, b]$ s.t.

$$\alpha = f(c)$$

Rolle's Theorem

f continuous on $[a, b]$,

differentiable on (a, b),

$f(a) = f(b)$

$\Rightarrow$ There exists $c \in (a, b)$

such that $f'(c) = 0$.

Cauchy's Mean Value Theorem

f continuous on $[a, b]$,

differentiable on (a, b)

Then there exists $c \in (a, b)$

such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Taylor's Theorem:

$f : [a, b] \longrightarrow \mathbb{R}$, $c \in [a, b]$,

$f, f', \dots, f^{(n)} \in C[a, b]$,

$f^{(n)}$: differentiable on (a, b)

Then for all $x \in [a, b]$,

$$f(x) = f(c) + f'(c)(x-c) + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \frac{f^{(n+1)}(d_x)}{(n+1)!}(x-c)^{n+1}$$

Chain Rule

$f : [a, b] \longrightarrow [\alpha, \beta]$, $g : [\alpha, \beta] \longrightarrow \mathbb{R}$

f : differentiable at c,

g : differentiable at $f(c)$.

Then

$$(g \circ f)'(c) = g'(f(c))\, f'(c)$$

Fundamental theorem of algebra

$p(x) = a_0 + a_1 x + \cdots + a_n x^n$,

where $a_0, a_1, \cdots, a_n$: real or complex numbers,

$a_n \neq 0$

Then $p(x)$ has at least one zero,

that is, $p(z) = 0$ for some $z \in \mathbb{C}$.

$f : [a, b] \longrightarrow \mathbb{R}$, $c \in (a, b)$

If $f(c) = 0$, but $f'(c) \neq 0$,

c is said to be a <u>simple zero</u> of f

If $f(c) = f'(c) = \cdots = f^{(m-1)}(c) = 0$,

but $f^{(m)}(c) \neq 0$, the c is said

to be a <u>zero of multiplicity m</u>.

f has a zero of multiplicity m at c : $f(c) = f'(c) = \cdots = f^{(m-1)}(c) = 0,$

$f^{(m)}(c) \neq 0$

Then $f(x) = (x-c)^m g(x),$

where $g(c) \neq 0$

$f(x) = \sin x$: 0 : simple zero

$f(x) = x \sin x$: 0 : multiplicity 2

Let p_n be a polynomial of degree n.
Then by the Fundamental theorem of algebra, $p_n(z_1) = 0$ for some $z_1 \in \mathbb{C}$.

$$p_n(x) = (x - z_1)\, q_{n-1}(x)$$

q_{n-1} : polynomial of degree $n-1$

$$p_n(x) = \alpha\, (x - z_1)^{m_1} \cdots (x - z_k)^{m_k},$$

$$m_1 + \cdots + m_k = n.$$

<u>Factorization</u> <u>Thm.</u>

A polynomial of degree n has exactly n zeroes, counted according to their multiplicities.

A non-zero polynomial of degree $\leq n$ has at most n distinct zeroes.

If a polynomial of degree $\leq n$ has <u>more than</u> n zeroes, then it is a zero polynomial.

Matrices

$A = [a_{ij}]$: $m \times n$ matrix, $x = [x(j)]$: $n \times 1$ vector

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad x = \begin{bmatrix} x(1) \\ x(2) \\ \vdots \\ x(n) \end{bmatrix}$$

Ax : $m \times 1$ vector

$$(Ax)(i) = a_{i1}\,x(1) + a_{i2}\,x(2) + \cdots + a_{in}\,x(n)$$

$$A = [a_{ij}] : m \times n$$

$$x = [x(j)] : n \times 1$$

$$(Ax)(i) = \sum_{j=1}^{n} a_{ij}\, x(j), \quad i = 1, \cdots, m$$

Ax : $m \times 1$ vector

$$A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$A(\alpha x + y) = \alpha Ax + Ay, \quad \alpha \in \mathbb{R}, \; x, y \in \mathbb{R}^n$$

$A = [a_{ij}] : m \times n$

$B = [b_{ij}] : n \times p$

$C = AB = [c_{ij}] : m \times p$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$A, B, C : n \times n$ $\quad (AB)C = A(BC)$

$AB \neq BA$