Note Title 3/12/2011

Numerical Integration: Basic Rules

Mean Value Theorem for Integrals

Let $f : [a, b] \to \mathbb{R}$ be continuous and let $g : [a, b] \to \mathbb{R}$ be an integrable function such that either $g(x) \geq 0$ or $g(x) \leq 0$, $x \in [a,b]$.

Then there exists $c \in [a,b]$ such that

$$\int_a^b f(x)\, g(x)\, dx = f(c) \int_a^b g(x)\, dx$$

$f : [a,b] \longrightarrow \mathbb{R}$,

p_n : interpolates f at $n+1$ points .

$$\int_a^b f(x)\,dx \simeq \int_a^b p_n(x)\,dx = \sum_{i=0}^{n} w_i\, f(x_i)$$

If f is a polynomial of degree $m \leq n$,

then $p_n(x) = f(x)$: $\displaystyle\int_a^b f(x)\,dx = \int_a^b p_n(x)\,dx$

<u>No</u> <u>error</u>

Quadrature rule is exact for polynomials of degree $\leq n$

$$f(x) = p_n(x) + \int_a^b f[x_0, x_1, \cdots, x_n, x]\, \omega(x)\, dx$$

$$\omega(x) = (x - x_0) \cdots (x - x_n)$$

$f[x_0, x_1, \cdots, x_n, x]$: Continuous on $[a, b]$

Note Title 3/10/2011

Rectangle Rule.

$f : [a,b] \longrightarrow \mathbb{R}$ be differentiable.

Then $f(x) = f(a) + f[a,x](x-a)$.

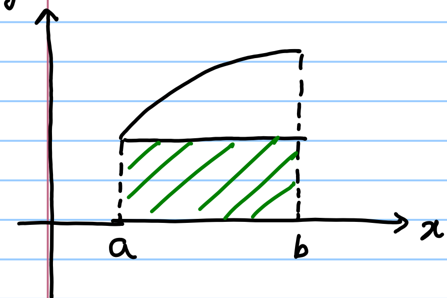
$$\int_a^b f(x)\,dx = \int_a^b f(a)\,dx + \int_a^b \underbrace{f[a,x]}_{\text{Continuous}} \underbrace{(x-a)}_{\geq 0}\,dx$$

By the MVT for integrals,

$$\int_a^b f(x)\,dx = f(a)(b-a) + f[a,c]\int_a^b (x-a)\,dx, \quad c \in [a,b]$$

$$= f(a)(b-a) + \underbrace{f[a,c]\,\frac{(b-a)^2}{2}}_{\text{error}}$$

$$\int_a^b f(x)\,dx \simeq f(a)(b-a)$$



$$\text{error} = f[a,c]\,\frac{(b-a)^2}{2}$$
$$= f'(\eta)\,\frac{(b-a)^2}{2}$$

If $f'(x) \equiv 0$, $x \in [a,b]$, that is, if f is a constant function, then the error is zero.

Rectangle Rule is exact for <u>constant functions</u>.

Continuity of $f[x_0, x_0, x]$

Let $f \in C^2[a,b]$, $x_0 \in [a,b]$.

$$f[x_0, x_0, x] = \begin{cases} \dfrac{f''(x_0)}{2}, & x = x_0 \\[2ex] \dfrac{f[x_0, x] - f'(x_0)}{x - x_0}, & x \neq x_0 \end{cases}$$

$$\lim_{x \to x_0} \frac{f[x_0, x] - f'(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{f(x) - f(x_0) - (x - x_0) f'(x_0)}{(x - x_0)^2},$$

$$= \lim_{x \to x_0} \frac{f''(c_x)(x - x_0)^2}{2(x - x_0)^2}, \quad \text{by the extended MVT}$$

$$= \frac{f''(x_0)}{2}$$

Midpoint Rule.

$f : [a, b] \longrightarrow \mathbb{R}$ twice differentiable.

$$f(x) = f\left(\frac{a+b}{2}\right) + f\left[\frac{a+b}{2}, x\right]\left(x - \frac{a+b}{2}\right)$$

$$\int_a^b f(x)\,dx = \int_a^b f\left(\frac{a+b}{2}\right)dx + \int_a^b f\left[\frac{a+b}{2}, x\right]\left(x - \frac{a+b}{2}\right)dx$$

$$= f\left(\frac{a+b}{2}\right)(b-a) + \int_a^b \underbrace{f\left[\frac{a+b}{2}, x\right]}_{\text{continuous}} \underbrace{\left(x - \frac{a+b}{2}\right)}_{\text{takes both +ve and -ve values}}dx$$

$$\boxed{\int_a^b \left(x - \frac{a+b}{2}\right)dx = 0}$$

$$\text{error} = \int_a^b f\left[\frac{a+b}{2}, x\right]\left(x - \frac{a+b}{2}\right) dx .$$

$$f\left[c, \frac{a+b}{2}, x\right] = \frac{f\left[\frac{a+b}{2}, x\right] - f\left[c, \frac{a+b}{2}\right]}{x - c}, \quad c \in [a,b]$$

$$f\left[\frac{a+b}{2}, x\right] = f\left[c \; \frac{a+b}{2}\right] + f\left[c \; \frac{a+b}{2} \; x\right](x - c), \quad x \in [a,b]$$

$$\text{error} = f\left[c, \frac{a+b}{2}\right] \underbrace{\int_a^b \left(x - \frac{a+b}{2}\right) dx}_{=0}$$

$$+ \int_a^b f\left[c, \frac{a+b}{2}, x\right](x-c)\left(x - \frac{a+b}{2}\right) dx .$$

$$\text{error} = \int_a^b f\left[c, \frac{a+b}{2}, x\right](x-c)\left(x-\frac{a+b}{2}\right)dx,$$

c any point in $[a,b]$

Choose $c = \frac{a+b}{2}$.

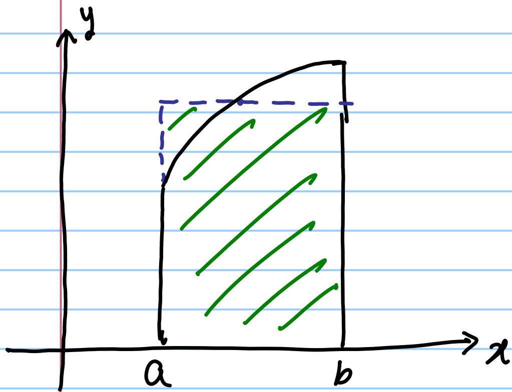
$$\text{error} = \int_a^b \underbrace{f\left[\frac{a+b}{2}, \frac{a+b}{2}, x\right]}_{\text{continuous}} \underbrace{\left(x-\frac{a+b}{2}\right)^2}_{\geq 0} dx$$

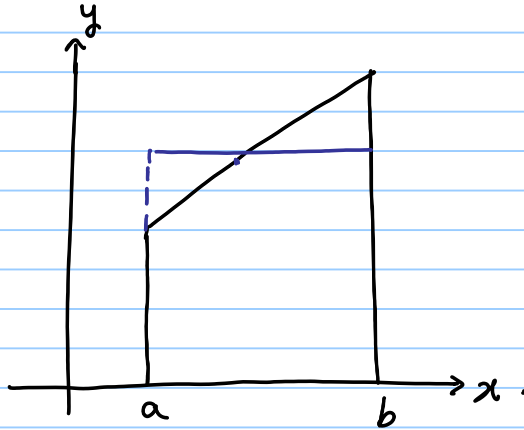
$$= f\left[\frac{a+b}{2}, \frac{a+b}{2}, d\right] 2\left(\frac{b-a}{2}\right)^3$$

$$= \frac{f''(\eta)}{24}(b-a)^3$$

exact for linear polynomials

Midpoint Rule





No error

Note Title 3/11/2011

Trapezoidal Rule.

$$f(x) = \underbrace{f(a) + f[a,b](x-a)}_{p_1(x)} + f[a,b,x](x-a)(x-b)$$

$p_1(a) = f(a)$, $p_1(b) = f(b)$

$$\int_a^b f(x)\,dx = \int_a^b p_1(x)\,dx + \int_a^b \underbrace{f[a,b,x]}_{\text{cont}^{s}} \underbrace{(x-a)(x-b)}_{\leq 0}\,dx$$

$$= f(a)(b-a) + f[a,b]\frac{(b-a)^2}{2}$$

$$+ f[a,b,c]\int_a^b (x-a)(x-b)\,dx$$

$$\int_a^b f(x)\,dx = f(a)(b-a) + f[a,b]\,\frac{(b-a)^2}{2}$$

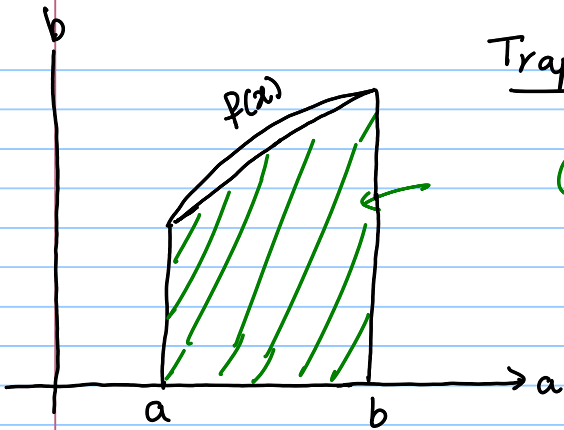
$$+ f[a,b,c]\int_a^b (x-a)(x-b)\,dx$$

$$= \frac{b-a}{2}\left(f(a)+f(b)\right) + f[a,b,c]\left\{ (x-b)\frac{(x-a)^2}{2}\Big|_a^b - \int_a^b \frac{(x-a)^2}{2}\,dx \right\}$$

$$\text{error} = -f[a,b,c]\,\frac{(b-a)^3}{6}$$

$$= -\frac{f''(\eta)}{12}(b-a)^3$$

<u>exact for linear polynomials</u>



Trapezoidal Rule

$$(b-a)\ \frac{f(a)+f(b)}{2}$$

Simpson Rule.

$$f(x) = \underbrace{f(a) + f[a,b](x-a) + f\left[a, b, \frac{a+b}{2}\right](x-a)(x-b)}_{p_2(x)}$$

$$+ f\left[a, b, \frac{a+b}{2}, x\right](x-a)(x-b)\left(x - \frac{a+b}{2}\right)$$

error

$$p_2(x) = f(a) + f[a,b](x-a) + f\left[a, b, \frac{a+b}{2}\right](x-a)(x-b)$$

$$\int_a^b p_2(x)\,dx = \frac{b-a}{2}\left(f(a)+f(b)\right) + f\left[a, \frac{a+b}{2}, b\right]\left\{-\frac{(b-a)^3}{6}\right\}$$

$$= \frac{b-a}{2}\left(f(a)+f(b)\right) + \frac{f(b) - 2f\left(\frac{a+b}{2}\right) + f(a)}{\frac{(b-a)^2}{2}}\left\{-\frac{(b-a)^3}{6}\right\}$$

$$= (b-a)\left[f(a)\left\{\frac{1}{2}-\frac{1}{3}\right\} + \frac{4}{6} f\left(\frac{a+b}{2}\right) + f(b)\left\{\frac{1}{2}-\frac{1}{3}\right\}\right]$$

$$= \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Error in Simpson Rule

$$\int_a^b f\left[a, b, \frac{a+b}{2}, x\right] \underbrace{(x-a)\left(x-\frac{a+b}{2}\right)(x-b)}_{\omega(x)} dx$$

$\omega(x) \geqslant 0, \quad x \in \left[a, \frac{a+b}{2}\right], \quad \omega(x) \leq 0, \quad x \in \left[\frac{a+b}{2}, x\right]$

$$\int_a^b (x-a)\left(x-\frac{a+b}{2}\right)(x-b)\,dx = \int_{-k}^{k} (y-k)\,y\,(y+k)\,dy$$

$$y = x - \frac{a+b}{2}, \quad k = \frac{b-a}{2}$$

$$= \int_{-k}^{k} (y^3 - k^2 y)\,dy = 0$$

$$\text{Error} = \int_a^b f\left[a, \frac{a+b}{2}, b, x\right] w(x)\,dx ,$$

$$\int_a^b w(x)\,dx = \int_a^b (x-a)\left(x-\frac{a+b}{2}\right)(x-b)\,dx = 0$$

$$f\left[a, \frac{a+b}{2}, b, x\right] = f\left[c, a, \frac{a+b}{2}, b\right] + f\left[c, a, \frac{a+b}{2}, b, x\right](x-c)$$

$$\text{Error} = f\left[c, a, \frac{a+b}{2}, b\right] \int_a^b w(x)\,dx = 0$$

$$+ \int_a^b f\left[c, a, \frac{a+b}{2}, b, x\right](x-c)\,w(x)\,dx$$

$$\text{Error} = \int_a^b f[c, a, \tfrac{a+b}{2}, b, x](x-c)\,\omega(x)\,dx$$

$$\omega(x) = (x-a)\left(x-\frac{a+b}{2}\right)(x-b).$$

$$c = \frac{a+b}{2} \Rightarrow \omega(x)(x-c) \le 0. \qquad k = \frac{b-a}{2}$$

$$\text{Error} = f[c, a, \tfrac{a+b}{2}, b, d] \int_a^b (x-a)\left(x-\frac{a+b}{2}\right)^2 (x-b)\,dx$$

$$= \frac{f^{(4)}(\eta)}{4!} \int_{-k}^{k} (y+k)\,y^2(y-k)\,dy = \frac{f^{(4)}(\eta)}{4!}\left[\frac{y^5}{5} - k^2\frac{y^3}{3}\right]_{-k}^{k}$$

$$= -\frac{f^{(4)}(\eta)}{90}\left(\frac{b-a}{2}\right)^5$$

exact for cubic polynomials

Rule	degree of the interpolating poly.	Exact for polynomials of degree
Rectangle	0	0
Midpoint	0	1
Trapezoidal	1	1
Simpson	2	3

Newton-Cotes formula

$$a = x_0 < x_1 < \cdots < x_n = b$$

$$h = \frac{b-a}{n}, \quad x_i = a + ih, \quad i = 0, 1, \cdots, n$$

$$p_n(x) = \sum_{i=0}^{n} f(x_i)\, l_i(x)$$

$$\int_a^b f(x)\,dx \simeq \int_a^b p_n(x)\,dx = \sum_{i=0}^{n} w_i\, f(x_i)$$

$n = 1$: Trapezoidal, $n = 2$: Simpson

$$n = 3 \;:\; \frac{3(b-a)}{8}\Big(f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\Big)$$

Rungé example: $f(x) = \dfrac{1}{1+25x^2}$, $x \in [-1, 1]$

$\|f - p_n\|_\infty \longrightarrow \infty$