Note Title 3/14/2011

Composite Numerical Quadrature

Rule	degree of the interpolating poly.	Exact for polynomials of degree
Rectangle	0	0
Midpoint	0	1
Trapezoidal	1	1
Simpson	2	3

Newton-Cotes formula

$$a = x_0 < x_1 < \cdots < x_n = b$$

$$h = \frac{b-a}{n}, \quad x_i = a + ih, \quad i = 0, 1, \cdots, n$$

$$p_n(x) = \sum_{i=0}^{n} f(x_i)\, l_i(x)$$

$$\int_a^b f(x)\,dx \simeq \int_a^b p_n(x)\,dx = \sum_{i=0}^{n} w_i\, f(x_i)$$

$n = 1$: Trapezoidal, $n = 2$: Simpson

$$n = 3 \; : \; \frac{3(b-a)}{8}\Big(f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\Big)$$

Rungé example: $f(x) = \dfrac{1}{1+25x^2}, \quad x \in [-1,1]$

$$\|f - p_n\|_\infty \longrightarrow \infty$$

Note Title 3/12/2011

Corrected Trapezoidal Rule.

$$f(x) = f(a) + f[a, b](x-a) + f[a, b, a](x-a)(x-b)$$
$$+ f[a, b, a, b](x-a)^2(x-b)$$
$$+ f[a, b, a, b, x](x-a)^2(x-b)^2.$$

$$= p_3(x) + f[a, a, b, b, x]\,\omega(x)$$

$$p_3(a) = f(a), \quad p_3'(a) = f'(a)$$
$$p_3(b) = f(b), \quad p_3'(b) = f'(b)$$

Cubic Hermite interpolation

$$\int_a^b P_3(x)\,dx = \int_a^b [f(a) + f[a,b](x-a)]\,dx$$

$$+ \int_a^b f[a,a,b](x-a)(x-b)\,dx +$$

$$\int_a^b f[a,a,b,b](x-a)^2(x-b)\,dx$$

$$= \frac{b-a}{2}\left(f(a)+f(b)\right) + f[a,a,b]\left\{-\frac{(b-a)^3}{6}\right\}$$

$$+ \frac{f[a,b,b] - f[a,a,b]}{b-a}\left\{(x-b)\frac{(x-a)^3}{3}\Big|_a^b - \int_a^b \frac{(x-a)^3}{3}\,dx\right\}$$

$$= \frac{b-a}{2}\left(f(a)+f(b)\right) - \left\{f\frac{[a,b]-f'(a)}{b-a} - \frac{f'(b)-f[a,b]}{12}\right\}\frac{(b-a)^3}{12}$$

$$\int_a^b p_3(x)\,dx = \frac{b-a}{2}\left(f(a)+f(b)\right) + \frac{(b-a)^2}{12}\left(f'(a)-f'(b)\right)$$

$$\text{error} = \int_a^b \underbrace{f[a,a,b,b,x]}_{\text{cont}^s} \underbrace{(x-a)^2(x-b)^2}_{\geq 0}\,dx$$

$$= f[a,a,b,b,c] \int_{-k}^{k} (y+k)^2(y-k)^2\,dy, \qquad \boxed{y = x - \frac{a+b}{2}, \quad k = \frac{b-a}{2}}$$

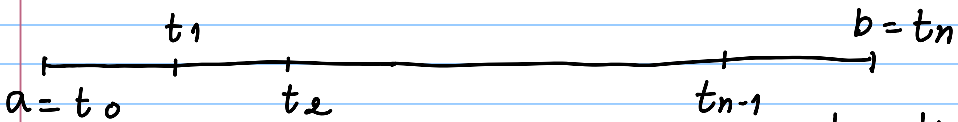
$$= \frac{f^{(4)}(\eta)}{24}\left[\frac{y^5}{5} - \frac{2k^2y^3}{3} + k^4 y\right]_{-k}^{k}$$

$$= \frac{f^{(4)}(\eta)}{24}\,\frac{16k^5}{15} = \frac{f^{(4)}(\eta)\,(b-a)^5}{720}$$

Composite Numerical Quadrature

Basic Rectangle Rule : $\int_a^b f(x)\,dx \simeq f(a)(b-a)$

Error $= f'(\eta)\dfrac{(b-a)^2}{2}$, $\eta \in [a,b]$



$$h = t_{i+1} - t_i = \frac{b-a}{n}$$

$$\int_{t_i}^{t_{i+1}} f(x)\,dx \simeq f(t_i)\,h$$

error $= f'(\eta_i)\dfrac{h^2}{2}$, $\eta_i \in [t_i, t_{i+1}]$

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n-1} \int_{t_{i-1}}^{t_i} f(x)\,dx \simeq \sum_{i=0}^{n-1} f(t_i)\,h$$

$$\text{error} = \sum_{i=0}^{n-1} f'(\eta_i)\,\frac{h^2}{2} = \frac{h^2}{2} \sum_{i=0}^{n-1} f'(\eta_i)$$

Assume that f' is continuous on $[a,b]$ and let $m = \min_{x\in[a,b]} f'(x)$, $M = \max_{x\in[a,b]} f'(x)$

$$m \leq f'(t_i) \leq M$$

$$\Rightarrow m \leq \frac{\sum_{i=0}^{n-1} f'(t_i)}{n} \leq M$$

By the Intermediate value theorem,

$$\frac{\sum_{i=0}^{n} f'(t_i)}{n} = f'(c) \text{ for some } c \in [a,b].$$

$$\text{error} = \frac{h^2}{2} \sum_{i=0}^{n} f'(t_i) = \left(\frac{b-a}{n}\right)^2 n \; f'(c) = (b-a) h f'(c)$$

Composite Rectangle Rule.

$$\int_a^b f(x)\,dx \simeq h \sum_{i=0}^{n-1} f(t_i) \qquad \text{error} = f'(c)(b-a)h.$$

Function evaluations: n, $f \in C^1[a,b]$

error $= O(h)$

Basic Midpoint Rule:

$$\int_a^b f(x)\,dx \simeq (b-a)\, f\left(\frac{a+b}{2}\right), \quad \text{error} = \frac{f''(\eta)(b-a)^3}{24}$$

<u>Composite</u> <u>Midpoint</u> <u>Rule</u>

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} f(x)\,dx \simeq h \sum_{i=0}^{n-1} f\left(\frac{t_i + t_{i+1}}{2}\right)$$

$$\text{Error} = \sum_{i=0}^{n-1} \frac{f''(\eta_i)\, h^3}{24} = \frac{h^2}{24}(b-a) \frac{\sum_{i=0}^{n-1} f''(\eta_i)}{n}$$

$$= \frac{(b-a)\, f''(\xi)}{24} h^2$$

function evaluations: n

$f \in C^2[a,b]$, error: $O(h^2)$

Trapezoidal Rule

$$\int_a^b f(x)\,dx \simeq \frac{b-a}{2}\left(f(a)+f(b)\right) \qquad \text{error} = -\frac{f''(\eta)}{12}(b-a)^3$$

Composite Trapezoidal Rule

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} f(x)\,dx \simeq \frac{h}{2}\sum_{i=0}^{n-1}\left[f(t_i)+f(t_{i+1})\right]$$

$$= \frac{h}{2}\left(f(a)+f(b)\right) + h\sum_{i=1}^{n-1} f(t_i)$$

$$\text{error} = -\frac{h^3}{12}\sum_{i=0}^{n-1} f''(\eta_i) = -\frac{h^2(b-a)}{12}\,\frac{\sum_{i=0}^{n-1} f''(\eta_i)}{n}$$

$$= -\frac{(b-a)f''(\xi)}{12}h^2$$

Function evaluations : $n+1$

$f \in C^2[a,b]$

error : $O(h^2)$

Simpson Rule

$$\int_a^b f(x)\,dx \simeq \frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)$$

$$\text{error} = -\frac{f^{(4)}(\eta_i)}{90}\left(\frac{b-a}{2}\right)^5$$

Composite Simpson Rule

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} f(x)\,dx \simeq \frac{h}{6}\sum_{i=0}^{n-1}\left[f(t_i)+4f(s_i)+f(t_{i+1})\right],$$

$$s_i = \frac{t_i+t_{i+1}}{2}$$

$$= \frac{h}{6}\left(f(a)+f(b)\right)+\frac{h}{3}\sum_{i=1}^{n-1} f(t_i)$$

$$+\frac{2h}{3}\sum_{i=0}^{n-1} f(s_i)$$

$$\text{error} = -\frac{1}{90}\left(\frac{h}{2}\right)^5 \sum_{i=0}^{n-1} f^{(4)}(\eta_i)$$

$$= -\frac{(b-a)\, f^{(4)}(\xi)}{180}\left(\frac{h}{2}\right)^4$$

function evaluations : $2n+1$

$f \in C^4[a,b]$

error : $O(h^4)$

Corrected Trapezoidal Rule.

$$\int_a^b f(x)\,dx \simeq \frac{b-a}{2}\left(f(a)+f(b)\right) + \frac{(b-a)^2}{12}\left(f'(a)-f'(b)\right)$$

$$\text{error} = \frac{f^{(4)}(\eta)}{720}(b-a)^5$$

Composite Corrected Trapezoidal Rule.

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} f(x)\,dx$$

$$\simeq \frac{h}{2}\sum_{i=0}^{n-1}\left(f(t_i)+f(t_{i+1})\right) + \frac{h^2}{12}\sum_{i=0}^{n-1}\left(f'(t_i)-f'(t_{i+1})\right)$$

$$= \frac{h}{2}\left(f(a)+f(b)\right) + h\sum_{i=1}^{n-1} f(t_i) + \frac{h^2}{12}\left(f'(a)-f'(b)\right)$$

$$\text{error} = \frac{h^5}{720} \sum_{i=0}^{n-1} f^{(4)}(\eta_i)$$

$$= \frac{(b-a)\, f^{(4)}(\xi)}{720}\, h^4$$

Function evaluations : $n+1$ + 2 derivatives

$f \in C^4[a,b]$

Error : $O(h^4)$

Composite Rule	Smoothness required	function evaluations	Order of convergence
Rectangle	C^1	n	h
Midpoint	C^2	n	h^2
Trapezoidal	C^2	$n+1$	h^2
Simpson	C^4	$2n+1$	h^4
Corrected Trapezoidal	C^4	$n+1+$ 2 derivatives	h^4