Note Title 3/16/2011

Gaussian Integration

So far we have fixed the interpolation points $x_0, x_1, \cdots, x_n$ in $[a,b]$, constructed the interpolating polynomial

$$p_n(x) = \sum_{i=0}^{n} f(x_i)\, l_i(x),$$

where $l_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{(x-x_j)}{(x_i-x_j)}$: Lagrange polynomial

and $$\int_a^b f(x)\,dx \simeq \sum_{i=0}^{n} f(x_i) \underbrace{\int_a^b l_i(x)\,dx}_{=w_i} = \sum_{i=0}^{n} w_i f(x_i)$$

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Starting Point:

$$\int_a^b f(x)\,dx \simeq \sum_{i=0}^{n} w_i\, f(x_i)$$

Question: Is it possible to determine real numbers w_i and points x_i in $[a,b]$ so that the error is zero for polynomials of degree $\leq 2n+1$?

Let $n = 0$. $\displaystyle\int_a^b f(x)\,dx \simeq w_0 f(x_0)$

To determine w_0 and $x_0 \in [a,b]$ such that error is zero for linear polynomials.

$\Leftarrow$ $f(x) = 1$. $b - a = w_0$

$f(x) = x$ $\dfrac{b^2 - a^2}{2} = w_0 x_0 \Rightarrow x_0 = \dfrac{a+b}{2}$.

$\displaystyle\int_a^b f(x)\,dx \simeq (b-a) f\left(\frac{a+b}{2}\right)$: <u>Midpoint Rule</u>.

Let $n = 1$. $\int_a^b f(x)\,dx \simeq w_0 f(x_0) + w_1 f(x_1)$

To determine w_0, w_1, x_0, x_1 such that there is no error for polynomials of degree ≤ 3.

$\Leftrightarrow$

$f(x) = 1$: $b - a = w_0 + w_1$

$f(x) = x$: $\dfrac{b^2 - a^2}{2} = w_0 x_0 + w_1 x_1$

$f(x) = x^2$: $\dfrac{b^3 - a^3}{3} = w_0 x_0^2 + w_1 x_1^2$

$f(x) = x^3$: $\dfrac{b^4 - a^4}{4} = w_0 x_0^3 + w_1 x_1^3$

Non-linear equations

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Problem:

To determine w_0, w_1, x_0, x_1 such that

$$\int_a^b f(x)\,dx \simeq w_0 f(x_0) + w_1 f(x_1)$$

is exact for polynomials of degree ≤ 3.

$$f(x) = f(x_0) + f[x_0, x_1](x - x_0)$$

$$+ f[x_0, x_1, x](x - x_0)(x - x_1)$$

$$\int_a^b f(x)\,dx = \int_a^b \left\{ f(x_0) + f[x_0, x_1](x - x_0) \right\} dx$$

$$+ \int_a^b f[x_0, x_1, x](x - x_0)(x - x_1)\,dx .$$

x_0, x_1 : any points in $[a, b] \Longrightarrow$

error $= 0$ for polynomials of degree ≤ 1

$$\text{error} = \int_a^b f[x_0, x_1, x] \overbrace{(x-x_0)(x-x_1)}^{\omega(x)} \, dx .$$

Suppose that x_0 and x_1 are such that

$$\int_a^b (x-x_0)(x-x_1)\, dx = 0 .$$

Then write $f[x_0, x_1, x] = f[y_0\ x_0\ x_1] +$

$$f[y_0, x_0, x_1, x](x-y_0)$$

$$\text{error} = f[y_0, x_0, x_1] \underbrace{\int_a^b \omega(x)\, dx}_{\overset{\shortparallel}{0}} + \int_a^b f[y_0, x_0, x_1, x] (x-y_0)\, \omega(x)\, dx$$

If $\int_a^b (x-x_0)(x-x_1)\,dx = 0$, then

error $= \int_a^b f[y_0, x_0, x_1, x]\,(x-y_0)(x-x_0)(x-x_1)\,dx$

Note that if f is a quadratic polynomial, then $f[y_0, x_0, x_1, x] = 0$.

Hence error $= 0$.

If $\int_a^b (x-x_0)(x-x_1)\,dx = 0$, then

error $= \int_a^b f[y_0, x_0, x_1, x]\,(x-y_0)(x-x_0)(x-x_1)\,dx$

If, in addition, $\int_a^b (x-x_0)(x-x_1)\,x\,dx = 0$, then

error $= \int_a^b f[y_1, y_0, x_0, x_1, x]\,(x-y_0)(x-y_1)(x-x_0)(x-x_1)\,dx$

$= 0$ if f is a cubic polynomial.

If $x_0, x_1 \in [a,b]$ are such that

$$\int_a^b (x-x_0)(x-x_1)\,dx = 0, \quad \int_a^b (x-x_0)(x-x_1)\,x\,dx = 0,$$

then

$$\int_a^b f(x)\,dx = \int_a^b \left[f(x_0) + f[x_0,x_1](x-x_0)\right] dx$$

$$+ \int_a^b f[x_0,x_0,x_1,x_1,x](x-x_0)^2(x-x_1)^2\,dx.$$

$$= w_0 f(x_0) + w_1 f(x_0) + \frac{f^{(4)}(d)}{4!} \int_a^b (x-x_0)^2(x-x_1)^2\,dx$$

Thus, the problem of finding $\omega_0, \omega_1, x_0, x_1$ such that

$$\int_a^b f(x)\,dx \simeq \omega_0 f(x_0) + \omega_1 f(x_1)$$

is exact for cubic polynomials reduces to finding x_0, x_1 such that

$$\int_a^b (x-x_0)(x-x_1)\,dx = 0, \quad \int_a^b (x-x_0)(x-x_1)\,x\,dx = 0$$

and putting $\omega_0 = \int_a^b \frac{x-x_1}{x_0-x_1}\,dx$, $\omega_1 = \int_a^b \frac{x-x_0}{x_1-x_0}\,dx$.

Inner Product

Let $f, g \in C[a,b]$. Define

$$\langle f, g \rangle = \int_a^b f(x)\, g(x)\, dx .$$

Then

1) $\langle f, f \rangle \geq 0$, $\quad \langle f, f \rangle = 0 \iff f(x) \equiv 0$.

2) $\langle f, g \rangle = \langle g, f \rangle$

3) $\langle f_1 + f_2, g \rangle = \langle f_1, g \rangle + \langle f_2, g \rangle$,

$\langle \alpha f, g \rangle = \alpha \langle f, g \rangle$, $\alpha \in \mathbb{R}$

Definition: $g_1, g_2, \cdots, g_n \in C[a,b]$ are said to be orthogonal if

$$\langle g_i, g_j \rangle = 0 , \quad i \neq j .$$

This is denoted by $g_i \perp g_j$, $i \neq j$

Example: $g_1(x) = 1$, $g_2(x) = x - \frac{a+b}{2}$.

$$\langle g_1, g_2 \rangle = \int_a^b \left(x - \frac{a+b}{2}\right) dx = 0$$

Definition: $g_1, g_2, \cdots, g_n \in C[a,b]$ are said to be orthonormal if

$$\langle g_i, g_j \rangle = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$$

Example:

$$g_1(x) = \frac{1}{\sqrt{b-a}}, \quad g_2(x) = 2\,\frac{x - \frac{a+b}{2}}{(b-a)\sqrt{b-a}}$$

Recall that we are trying to find x_0, x_1 in $[a,b]$ such that

1) $$\int_a^b (x-x_0)(x-x_1)\,dx = 0$$

2) $$\int_a^b \underbrace{(x-x_0)(x-x_1)}_{p_2(x)}\, x\,dx = 0 .$$

Thus we want to find $p_2(x)$ such that $p_2 \perp f_0(x) \equiv 1$ and $p_2 \perp f_1(x) = x$.

Gram-Schmidt Orthonormalization

Consider $f_0(x) = 1$, $f_1(x) = x$, $f_2(x) = x^2$.

Define $g_0(x) = \dfrac{f_0(x)}{\|f_0\|}$: Constant Polynomial,

$r_1(x) = f_1(x) - \langle f_1, g_0 \rangle g_0(x) \Rightarrow \langle r_1, g_0 \rangle = 0$

$g_1(x) = \dfrac{r_1(x)}{\|r_1\|}$: linear polynomial,

$r_2(x) = f_2(x) - \langle f_2, g_0 \rangle g_0(x) - \langle f_2, g_1 \rangle g_1(x)$

$g_2(x) = \dfrac{r_2(x)}{\|r_2\|}$: quadratic, $\langle g_2, g_1 \rangle = \langle g_2, g_0 \rangle = 0$

Let $[a,b] = [-1,1]$.

$f_0(x) = 1 \Rightarrow \|f_0\| = \left(\int_{-1}^{1} dx\right)^{1/2} = \sqrt{2} \Rightarrow g_0(x) = \frac{1}{\sqrt{2}}$

$f_1(x) = x$, $r_1(x) = x - \langle f_1, g_0 \rangle g_0 = x$,

since $\langle f_1, g_0 \rangle = \int_{-1}^{1} \frac{x}{\sqrt{2}} = 0$.

$\|r_1\| = \left(\int_{-1}^{1} x^2 dx\right)^{1/2} = \sqrt{\frac{2}{3}}$, $g_1(x) = \sqrt{\frac{3}{2}}\, x$

$$g_0(x) = \frac{1}{\sqrt{2}}, \quad g_1(x) = \sqrt{\frac{3}{2}}\, x,$$

$$\|g_0\|_2 = \|g_1\|_2 = 1, \qquad \langle g_0, g_1 \rangle = 0 .$$

$$f_2(x) = x^2 \implies \langle f_2, g_0 \rangle = \int_{-1}^{1} \frac{x^2}{\sqrt{2}}\, dx = \frac{\sqrt{2}}{3}$$

$$\langle f_2, g_1 \rangle = \int_{-1}^{1} \sqrt{\frac{3}{2}}\, x^3\, dx = 0$$

$$r_2(x) = x^2 - \frac{\sqrt{2}}{3} \cdot \frac{1}{\sqrt{2}} = x^2 - \frac{1}{3}$$

$$\langle r_2, g_0 \rangle = \langle r_2, g_1 \rangle = 0$$

$$g_0(x) = \frac{1}{\sqrt{2}}, \quad g_1(x) = \sqrt{\frac{3}{2}}\, x,$$

$$r_2(x) = x^2 - \frac{1}{3}, \quad \langle r_2, g_0 \rangle = \langle r_2, g_1 \rangle = 0 .$$

Thus

$$\int_{-1}^{1} \left(x + \frac{1}{\sqrt{3}}\right)\left(x - \frac{1}{\sqrt{3}}\right) dx = 0, \quad \int_{-1}^{1} \left(x + \frac{1}{\sqrt{3}}\right)\left(x - \frac{1}{\sqrt{3}}\right) x\, dx = 0$$

$$x_0 = -\frac{1}{\sqrt{3}}, \quad x_1 = \frac{1}{\sqrt{3}} :$$

Gauss points in $[-1, 1]$