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Legendre Polynomials

$X = C[a,b]$, $\langle f, g \rangle = \int_a^b f(x) g(x)\, dx$, $\|f\|_2 = \sqrt{\langle f, f \rangle}$

$f_0(x) = 1$, $f_1(x) = x$, ... , $f_n(x) = x^n$, ...

Gram-Schmidt Orthonormalization

$g_0(x) = \frac{f_0}{\|f_0\|}$

span $\{f_0, f_1, \ldots, f_n\}$ = span $\{g_0, \ldots, g_n\}$

for $n = 1, 2, \ldots$

$r_n = f_n - \sum_{j=0}^{n-1} \langle f_n, g_j \rangle g_j$, $g_n = \frac{r_n}{\|r_n\|_2}$: poly. of degree n

$g_0, g_1, g_2, \cdots, g_n, \cdots$: <u>Legendre</u> <u>Polynomials</u>.

g_n : polynomial of degree n,

$$\langle g_i, g_j \rangle = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

$$\langle g_{n+1}, g_j \rangle = 0 \quad \text{for } j = 0, 1, \cdots, n$$

$$\text{span}\{g_0, \cdots, g_n\} = \text{span}\{1, x, \cdots, x^n\}$$

$$\langle g_{n+1}, a_0 + a_1 x + \cdots + a_n x^n \rangle = 0$$

g_{n+1} has $n+1$ distinct zeros, say, $x_0, x_1, \cdots, x_n$

Gauss Points

$$g_{n+1}(x) = \alpha_{n+1}(x-x_0)(x-x_1)\cdots(x-x_n)$$

$$\langle g_{n+1}, a_0 + a_1 x + \cdots + a_n x^n \rangle = 0$$

$$\boxed{\int_a^b (x-x_0)(x-x_1)\cdots(x-x_n)\, x^j \, dx = 0 \quad \text{for } j = 0, 1, \cdots, n}$$

Let $f \in C[a,b]$ and $p_n(x) = \sum_{i=0}^{n} f(x_i) l_i(x)$.

$$p_n(x_j) = f(x_j), \quad j = 0, 1, \cdots, n.$$

We have

$$f(x) - p_n(x) = f[x_0, x_1, \cdots, x_n, x]\, \omega(x),$$

where $\omega(x) = (x - x_0) \cdots (x - x_n)$.

$$\int_a^b \omega(x)\, x^j = 0, \quad j = 0, 1, \cdots, n$$

$$\int_a^b f(x)\,dx - \int_a^b p_n(x)\,dx = \int_a^b f[x_0, x_1, \cdots, x_n, x]\, \omega(x)\,dx$$

$$f[x_0, x_1, \cdots, x_n, x] = f[x_0, x_0, x_1, \cdots, x_n]$$

$$+ f[x_0, x_0, x_1, x_1, x_2, \cdots, x_n](x-x_0)$$

$$+ f[x_0, x_0, x_1, x_1, x_2, x_2, x_3, \cdots, x_n](x-x_0)(x-x_1)$$

$$+ \cdots + f[x_0, x_0, \cdots, x_n, x_n, x]\underbrace{(x-x_0)\cdots(x-x_n)}_{\omega(x)}.$$

$$\int_a^b \omega(x)\, x^j = 0, \quad j = 0, 1, \cdots, n$$

$$\int_a^b f[x_0, x_1, \cdots, x_n, x]\, \omega(x)\, dx$$

$$= \int_a^b f[x_0, x_0, \cdots, x_n, x_n, x]\, \omega(x)^2 dx$$

$$\int_a^b f(x)\,dx = \int_a^b \sum_{i=0}^{n} f(x_i)\, l_i(x) + \int_a^b f[x_0, x_1, \cdots, x_n, x]\, \omega(x)\, dx$$

$$= \sum_{i=0}^{n} f(x_i) \int_a^b l_i(x)\, dx + \int_a^b f[x_0, x_0, \cdots, x_n, x_n, x]\, \omega(x)^2\, dx$$

$$= \sum_{i=0}^{n} \omega_i\, f(x_i) + f[x_0, x_0, \cdots, x_n, x_n, x] \int_a^b \omega(x)^2\, dx$$

$$= \sum_{i=0}^{n} \omega_i\, f(x_i) + \frac{f^{(2n+2)}(c)}{(2n+2)!} \int_a^b (x-x_0)^2 \cdots (x-x_n)^2\, dx$$

Integration at Gauss points

$$\int_a^b f(x)\,dx - \sum_{i=0}^{n} w_i\, f(x_i)$$

exact for polys. of degree $\leq 2n+1$

$$= \frac{f^{(2n+2)}(c)}{(2n+2)!} \int_a^b (x-x_0)^2 \cdots (x-x_n)^2\, dx$$

$$\left| \int_a^b f(x)\,dx - \int_a^b p_n(x)\,dx \right| \leq C \,\| f^{(2n+2)} \|_\infty \,(b-a)^{2n+3}$$

Weights in Gaussian Integration

$x_0, x_1, \cdots, x_n$: Gauss points

$$\int_a^b f(x)\,dx \simeq \sum_{i=0}^{n} w_i\, f(x_i)$$

Claim: $w_i > 0$

Note that $\omega_i = \int_a^b l_i(x)\,dx$

Since $\sum_{j=0}^{n} l_j(x) = 1$, $\quad \omega_i = \int_a^b l_i(x)\left(\sum_{j=0}^{n} l_j(x)\right)dx$.

Thus

$$\omega_i = \int_a^b l_i^2(x)\,dx + \sum_{\substack{j=0\\ j\neq i}}^{n} \int_a^b l_i(x)\,l_j(x)\,dx.$$

Consider for $i \neq j$,

$$l_i(x)\, l_j(x) = \prod_{\substack{k=0\\k\neq i}}^{n} \frac{(x-x_k)}{(x_i-x_k)} \prod_{\substack{l=0\\l\neq j}}^{n} \frac{(x-x_l)}{(x_j-x_l)}$$

$$= \frac{(x-x_0)\cdots(x-x_n)}{\left[\prod\limits_{\substack{k=0\\k\neq i}}^{n}(x_i-x_k)\right](x_j-x_i)} \prod_{\substack{l=0\\l\neq j\\l\neq i}}^{n} \frac{(x-x_l)}{(x_j-x_l)}$$

$$= \frac{w(x)}{c}\, q_{n-1}(x), \quad q_{n-1} : \text{polynomial of degree } n-1$$

Thus for $i \neq j$,

$$\int_a^b l_i(x)\, l_j(x)\, dx = \frac{1}{C} \int_a^b \omega(x)\, q_{n-1}(x)\, dx$$

$$= 0 \quad ,$$

Since $x_0, x_1, \dots, x_n$ are Gauss points.

As $l_i(x)$ is a polynomial of degree n,

$\int_a^b l_i^2(x)\, dx > 0$. Hence $\omega_i > 0$.

Convergence of Gaussian Integration

Let $f \in C[a,b]$.

Denote $I_n(f) = \sum_{i=0}^{n} w_i^{(n)} f(x_i^{(n)})$: Gaussian Integration.

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n} w_i^{(n)} f(x_i^{(n)})$$ if f is a polynomial of degree $\le 2n+1$

In particular, $\sum_{i=0}^{n} w_i^{(n)} = b-a$.

Claim: $I_n(f) \longrightarrow \int_a^b f(x)\,dx$ as $n \to \infty$.

Fix $\epsilon > 0$. Since $f \in C[a,b]$, by the Weierstrass theorem, there exists a polynomial of degree $\le m$ such that $\|f - q_m\|_\infty < \epsilon$.

$$\int_a^b q_m(x)\,dx = \sum_{j=0}^{n} w_j^{(n)} q_m(x_j^{(n)}) = I_n(q_m)$$

for $n \geq \frac{m-1}{2}$.

Consider

$$\left| \int_a^b f(x)\,dx - I_n(f) \right|$$

$$= \left| \int_a^b f(x)\,dx - \int_a^b q_m(x)\,dx + I_n(q_m) - I_n(f) \right|$$

$$\leq \int_a^b |f(x) - q_m(x)|\,dx + \sum_{j=0}^{n} \omega_j^{(n)} \left| q_m(x_j^{(n)}) - f(x_j^{(n)}) \right|$$

$$\leq \|f - q_m\|_\infty (b-a) + \|f - q_m\|_\infty (b-a) < 2\epsilon(b-a)$$

for $n \geq \frac{m-1}{2}$

Thus $I_n(f) \longrightarrow \int_a^b f(x)\,dx$

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Newton - Cotes formulae.

$x_0, x_1, \cdots, x_n$: equidistant points.

$h = \frac{b-a}{n}$, $x_i^{(n)} = a + ih$, $i = 0, 1, \cdots, n$: $x_0 = a$, $x_n = b$.

$$p_n(x) = \sum_{i=0}^{n} f(x_i^{(n)})\, l_i^{(n)}(x)$$

$$\int_a^b f(x)\,dx \simeq \int_a^b \sum_{i=0}^{n} f(x_i^{(n)})\, l_i^{(n)}(x)\,dx$$

$$= \underbrace{\sum_{i=0}^{n} \omega_i^{(n)} f(x_i^{(n)})}_{I_n(f)}$$: exact for polys. of degree $\leq n$

Let $f \in C[a,b]$. Fix $\epsilon > 0$.

By the Weierstrass Theorem, there exists a polynomial q_m of degree $\leq m$ such that

$$\| f - q_m \|_\infty < \epsilon .$$

$$\int_a^b q_m(x)\,dx = I_n(q_m) \text{ for } n \geq m .$$

$$\left| \int_a^b f(x)\,dx - I_n(f) \right| \leq \left| \int_a^b f(x)\,dx - \int_a^b q_m(x)\,dx \right| + \left| I_n(q_m) - I_n(f) \right|$$

$$\left| \int_a^b f(x)\,dx - \int_a^b q_m(x)\,dx \right| \le \int_a^b |f(x) - q_m(x)|\,dx$$

$$\le \|f - q_m\|_\infty (b-a)$$

$$= \varepsilon (b-a)$$

$$\left| I_n(f) - I_n(q_m) \right| = \left| \sum_{j=0}^{n} w_j^{(n)} f(x_j^{(n)}) - \sum_{j=0}^{n} w_j^{(n)} q_m(x_j^{(n)}) \right|$$

$$\le \sum_{j=0}^{n} |w_j^{(n)}| \, |f(x_j^{(n)}) - q_m(x_j^{(n)})|$$

$$\le \epsilon \left(\sum_{j=0}^{n} |w_j^{(n)}| \right)$$

not bounded.

Gaussian Integration

1) fast convergence.

2) interpolation points : irrational

3) The interpolation points for $I_n(f)$ are different than the preceeding interpolation points for $I_m(f)$, $m < n$