Tutorial 2

Note Title 3/21/2011

Q.1 Let
$$f(x) = \begin{cases} x, & 0 \le x \le 1/2 \\ 1-x, & \frac{1}{2} \le x \le 1. \end{cases}$$
Find $\int_0^1 f(x)\,dx$ using

1) Trapezoidal Rule :

2) Composite Trapezoidal Rule on $0 < \frac{1}{2} < 1$

3) Simpson Rule, 4) Corrected Trapezoidal Rule.

$f(0) = f(1) = 0, \quad f\left(\frac{1}{2}\right) = \frac{1}{2}, \quad f'(0) = 1, \quad f'(1) = -1$

Note Title 3/21/2011

Q.1 Exact Value $= \int_0^{1/2} x \, dx + \int_{1/2}^{1} (1-x) \, dx = \frac{1}{8} + \frac{1}{2} - \left(\frac{1}{2} - \frac{1}{8}\right)$

$= \frac{1}{4}$

1) Trapezoidal Rule : $\frac{(b-a)}{2}(f(a) + f(b)) = 0$

2) Composite Trapezoidal Rule on $0 < \frac{1}{2} < 1$:

$\frac{1}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{4}$

3) Simpson Rule : $\frac{(b-a)}{6}\left\{f(a) + 4 f\left(\frac{a+b}{2}\right) + f(b)\right\} = \frac{1}{3}$

4) Corrected Trapezoidal Rule :

$\frac{1}{12}(f'(1) - f'(0)) = -\frac{1}{6}$

Q. 2 Construct a rule of the form

$$\int_{-1}^{1} f(x)\,dx \simeq A_0 f(-\tfrac{1}{2}) + A_1 f(0) + A_2 f(\tfrac{1}{2})$$

which is exact for polynomials of degree ≤ 2.

$f(x) = 1 \implies 2 = A_0 + A_1 + A_2$

$f(x) = x \implies 0 = -\frac{A_0}{2} + \frac{A_2}{2} \implies A_2 = A_0$

$f(x) = x^2 \implies \frac{2}{3} = \frac{A_0}{4} + \frac{A_2}{4} \implies A_0 = A_2 = \frac{4}{3}$

$\implies A_1 = 2 - \frac{8}{3} = -\frac{2}{3}$

Q.3. Let $x_0 \in \mathbb{R}$ and $x_k = x_0 + kh$, $k = 1, 2, 3$.

$$\int_{x_0}^{x_3} f(x)\,dx \simeq \frac{3h}{8}\left(f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right)$$

Determine the degree of precision of this rule.

<u>Solution</u>: Without loss of generality, let $x_0 = 0$.

$$\int_{0}^{3h} f(x)\,dx \simeq \frac{3h}{8}\left(f(0) + 3f(h) + 3f(2h) + f(3h)\right)$$

$$\int_0^{3h} f(x)\,dx \simeq \frac{3h}{8}\left(f(0)+3f(h)+3f(2h)+f(3h)\right)$$

$f(x)=1$: $LHS = 3h$, $RHS = \frac{3h}{8}(8) = 3h$

$f(x)=x$: $LHS = \frac{9h^2}{2}$, $RHS = \frac{3h}{8}(3h+6h+3h) = \frac{9h^2}{2}$.

$f(x)=x^2$: $LHS = \frac{27h^3}{3}$, $RHS = \frac{3h}{8}(3h^2+12h^2+9h^2) = 9h^3$

$f(x)=x^3$: $LHS = \frac{81h^4}{4}$, $RHS = \frac{3h}{8}(3h^3+24h^3+27h^3) = \frac{81h^4}{4}$

$f(x)=x^4$: $LHS = \frac{243h^5}{5} \neq RHS = \frac{99h^5}{2}$

degree of precision = 3

Q.4 Consider approximation of $\int_1^7 \frac{dx}{x}$ by composite Trapezoidal rule with step size h. Determine h and number of intervals so that the error is less that 4×10^{-8}

Solution : Error $= -\frac{f''(c)}{12} h^2 (b-a)$, $h = \frac{b-a}{n}$, $c \in [a,b]$.

$f(x) = \frac{1}{x}$, $f'(x) = -\frac{1}{x^2}$,

$f''(x) = \frac{2}{x^3} \Rightarrow \|f''\|_\infty \leq 2$

$$|\text{error}| \le \frac{\|f''\|_\infty}{12} \, 6 \, h^2 = h^2 < 4 \times 10^{-8}$$

$$\Rightarrow h < 2 \times 10^{-4}, \quad h = \frac{b-a}{n} = \frac{6}{n}.$$

$$\Rightarrow \frac{6}{n} < 2 \times 10^{-4} \Rightarrow n > 3 \times 10^{4} = 30000$$

Composite Simpson : $$|\text{error}| \le \frac{\|f^{(4)}\|_\infty \left(\frac{h}{2}\right)^4 (b-a)}{180}$$

$$f''(x) = \frac{2}{x^3}, \quad f^{(4)}(x) = \frac{24}{x^5}$$

$$\le \frac{24 \times 6 \, h^4}{16 \times 180} = \frac{h^4}{20}$$

$$h^4 < 80 \times 10^{-8}$$

$$\Rightarrow h < (80)^{1/4} \, 10^{-2} < 3 \times 10^{-2} \qquad < 4 \times 10^{-8}$$

$$\Rightarrow h > 2 \times 10^{2} = 200$$

Note Title 1/31/2011

Composite Corrected Trapezoidal Rule

$$\int_a^b f(x)\,dx = \frac{h}{2}\left(f(a)+f(b)\right) + h\sum_{i=1}^{n-1} f(t_i)$$

$$+ \frac{h^2}{12}\left(f'(a)-f'(b)\right) + O(h^4)$$

$$= T_n + C_1 h^2 + O(h^4) \qquad \cdots (1)$$

Note Title 1/31/2011

$$\int_a^b f(x)\,dx = T_n + C_1 h^2 + O(h^4) \quad \cdots (1)$$

$$\int_a^b f(x)\,dx = T_{\frac{n}{2}} + C_1 (2h)^2 + O(h^4) \quad \cdots (2)$$

$$\int_a^b f(x)\,dx = \frac{4T_n - T_{\frac{n}{2}}}{3} + O(h^4) = T_n^1 + O(h^4)$$

$f \in C^{2k+2}[a,b] \Rightarrow$

$$\int_a^b f(x)\,dx = T_n + C_1 h^2 + C_2 h^4 + \cdots + C_k h^{2k} + O(h^{2k+2})$$

C_i : independent of h (hence of n)

$$C_i = \alpha_i \left(f^{(k-1)}(a) - f^{(k-1)}(b) \right).$$

$$\int_a^b f(x)\,dx = T_n + C_1 h^2 + C_2 h^4 + O(h^6)$$

$$= T_{\frac{n}{2}} + C_1 (2h)^2 + C_2 (2h)^4 + O(h^6)$$

$$\int_a^b f(x)\,dx = T_n^1 + \frac{C_2 h^4 (4-16)}{3} + O(h^6)$$

$$= T_n^1 + C_2' h^4 + O(h^6) \qquad C_2' = -9\,C_2$$

$$\int_a^b f(x)\,dx = T_n^1 + C_2' h^4 + O(h^6)$$

$$= T_{\frac{n}{2}}^1 + C_2' (2h)^4 + O(h^6)$$

$$\int_a^b f(x)\,dx = \frac{16\, T_n^1 - T_{\frac{n}{2}}^1}{15} + O(h^6).$$

$$T_n^2 = \frac{16\, T_n^1 - T_{\frac{n}{2}}^1}{15}$$

$$
\begin{tikzcd}
T_n \arrow[dr, "4"] & & \\
 & T_n^1 \arrow[dr, "16"] & \\
T_{\frac{n}{2}} \arrow[ur, "1"'] \arrow[dr, "4"] & & T_n^2 \\
 & T_{\frac{n}{2}}^1 \arrow[ur, "1"'] & \\
T_{\frac{n}{4}} \arrow[ur, "1"'] & &
\end{tikzcd}
$$

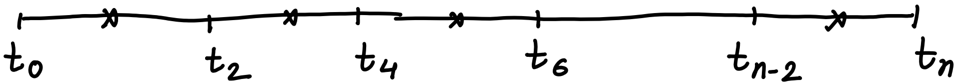
$$T_n^m = \frac{4^m \, T_n^{m-1} - T_{\frac{n}{2}}^{m-1}}{4^m - 1}, \quad T_n^0 = T_n$$

$$m = 1, 2, \cdots, k$$

Romberg Integration

$$\int_a^b f(x)\,dx = T_n^m + O\left(h^{2m+2}\right),$$

$$m = 0, 1, \cdots, k$$



$$T_n = \frac{h}{2}\left(f(a)+f(b)\right) + h\sum_{i=1}^{n-1} f(t_i), \quad h = \frac{b-a}{n}.$$

$$T_{\frac{n}{2}} = h\left(f(a)+f(b)\right) + 2h\sum_{\substack{i=2 \\ i \text{ even}}}^{n-2} f(t_i)$$

$$T_n^1 = \frac{4T_n - T_{\frac{n}{2}}}{3} = \frac{h}{3}\left(f(a)+f(b)\right) + \frac{2h}{3}\sum_{\substack{i=2 \\ i \text{ even}}}^{n-2} f(t_i)$$

$$+ \frac{4h}{3}\sum_{\substack{i=1 \\ i \text{ odd}}}^{n-1} f(t_i)$$

Simpson Rule

Numerical differentiation

$f : [c, d] \longrightarrow \mathbb{R}$, $a \in (c, d)$

Aim : To find an approximation of $f'(a)$

$f : [c, d] \longrightarrow \mathbb{R}$, $a \in (c, d)$

x_0, x_1 : distinct points in $[c, d]$

$$f(x) = \underbrace{f(x_0) + f[x_0\ x_1](x - x_0)}_{p_1(x)} + f[x_0\ x_1\ x]\underbrace{(x-x_0)(x-x_1)}_{\omega(x)}$$

$$f'(x) = p_1'(x) + \frac{d}{dx}\left\{ f[x_0\ x_1\ x]\ \omega(x) \right\}$$

$$= f[x_0\ x_1] + \left\{ \frac{d}{dx} f[x_0\ x_1\ x] \right\} \omega(x) + f[x_0\ x_1\ x]\ \omega'(x)$$

$$f'(x) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} + \frac{d}{dx}\{f[x_0\ x_1\ x]\}(x - x_0)(x - x_1)$$

$$+ f[x_0\ x_1\ x]\{x - x_1 + x - x_0\}$$

$x = a$, $x_0 = a$, $x_1 = a + h$

$$f'(a) = \frac{f(a+h) - f(a)}{h} + f[a\ \ a+h\ \ a](-h)$$

$$\text{error} = -\frac{f''(c)}{2} h$$

$$f'(x) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} + \frac{d}{dx}\left\{ f[x_0\, x_1\, x] \right\} (x - x_0)(x - x_1)$$

$$+ f[x_0\, x_1\, x] \left\{ x - x_1 + x - x_0 \right\}$$

$x = a, \quad x_0 = a - h, \quad x_1 = a + h$

$$f'(a) = \frac{f(a+h) - f(a-h)}{2h} + \frac{d}{dx}\left\{ f[a-h\;\, a+h\;\, x] \right\}\Big|_{x=a} (-h^2)$$

Derivative of $f[x_0 x]$

Let $g(x) = f[x_0 x] = \begin{cases} \dfrac{f(x) - f(x_0)}{x - x_0}, & x \neq x_0 \\ f'(x_0), & x = x_0 . \end{cases}$

For $x \neq x_0$,

$$g'(x) = \frac{(x - x_0) f'(x) - [f(x) - f(x_0)]}{(x - x_0)^2}$$

$$= \frac{f'(x) - f[x_0, x]}{x - x_0} = f[x_0, x, x]$$

Consider $\dfrac{g(x_0+h) - g(x_0)}{h}$

$$= \frac{f[x_0\ x_0+h] - f'(x_0)}{h} = \frac{f(x_0+h) - f(x_0) - h f'(x_0)}{h^2}$$

$$= \frac{\frac{h^2}{2} f''(c)}{h^2}, \quad c \in [x_0, x_0+h] \text{ or } [x_0+h, x_0] \ (h<0)$$

$$\longrightarrow \frac{f''(x_0)}{2} = f[x_0, x_0, x_0]$$

Thus $g'(x) = \frac{d}{dx}\left(f[x_0, x]\right) = f[x_0, x, x]$

Thus $\frac{d}{dx} f[x_0\ x] = f[x_0\ x\ x]$

Consider $f[x_0\ x_1\ x] = g(x)$, $x_0 \neq x_1$.

$$g(x) = f[x_0\ x\ x_1] = \frac{f[x\ x_1] - f[x_0\ x]}{x_1 - x_0}$$

$$g'(x) = \frac{f[x_1\ x\ x] - f[x_0\ x\ x]}{x_1 - x_0}$$

$$= f[x_0\ x_1\ x\ x]$$

$$g(x) = f[x_0\ x_1\ x_2\ x] = \frac{f[x_1\ x_2\ x] - f[x_0\ x_1\ x]}{x_2 - x_0}$$

$$g'(x) = \frac{f[x_1\ x_2\ x\ x] - f[x_0\ x_1\ x\ x]}{x_2 - x_0} = f[x_0\ x_1\ x_2\ x\ x]$$