Note Title 3/22/2011

Romberg Integration

$f \in C^4[a,b]$

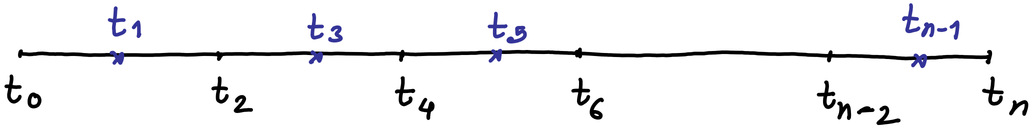
$a = t_0 < t_1 < \cdots < t_n = b$, $h = \frac{b-a}{n}$, n : even

$T_n = \frac{h}{2}\left(f(a)+f(b)\right) + h\sum_{i=1}^{n-1} f(t_i)$: Composite Trapezoidal Rule

$$\int_a^b f(x)\,dx = T_n + \frac{h^2}{12}\left(f'(a)-f'(b)\right) + O(h^4)$$

$$\int_a^b f(x)\,dx = T_{\frac{n}{2}} + \frac{(2h)^2}{12}\left(f'(a)-f'(b)\right) + O(h^4)$$

$$T_n^1 = \frac{4\,T_n - T_{\frac{n}{2}}}{3} \qquad \int_a^b f(x)\,dx = T_n^1 + O(h^4)$$



$$T_n = \frac{h}{2}\left(f(a) + f(b)\right) + h \sum_{i=1}^{n-1} f(t_i)$$

$$T_{\frac{n}{2}} = h\left(f(a) + f(b)\right) + 2h \sum_{\substack{i=2 \\ i \text{ even}}}^{n-2} f(t_i)$$

$$T_n^1 = \frac{4T_n - T_{\frac{n}{2}}}{3} = \frac{h}{3}\left(f(a) + f(b)\right) + \frac{4h}{3} \sum_{\substack{i=1 \\ i \text{ odd}}}^{n-1} f(t_i) + \frac{2h}{3} \sum_{\substack{i=2 \\ i \text{ even}}}^{n-1} f(t_i)$$

<u>Simpson Rule</u>

Numerical differentiation

$f : [c, d] \longrightarrow \mathbb{R}$, $a \in (c, d)$

Aim: To find an approximation to $f'(a)$

Approximate f by an interpolating polynomial p_n of degree $\leq n$ and then

$$f'(a) \simeq p_n'(a)$$

x_0, x_1 : distinct points in $[c, d]$

$$f(x) = \underbrace{f(x_0) + f[x_0\, x_1](x - x_0)}_{p_1(x)} + f[x_0\, x_1\, x]\underbrace{(x-x_0)(x-x_1)}_{\omega(x)}$$

$$f'(x) = p_1'(x) + \frac{d}{dx}\left\{ f[x_0\, x_1\, x]\, \omega(x) \right\}$$

$$= f[x_0\, x_1] + \left\{ \frac{d}{dx} f[x_0\, x_1\, x] \right\} \omega(x) + f[x_0\, x_1\, x]\, \omega'(x)$$

$$\boxed{\textbf{Claim: } \frac{d}{dx} f[x_0, x] = f[x_0, x, x].}$$

Proof:

Let $g(x) = f[x_0 x] = \begin{cases} \dfrac{f(x) - f(x_0)}{x - x_0}, & x \neq x_0 \\ f'(x_0), & x = x_0. \end{cases}$

For $x \neq x_0$,

$$g'(x) = \frac{(x - x_0) f'(x) - [f(x) - f(x_0)]}{(x - x_0)^2}$$

$$= \frac{f'(x) - f[x_0 x]}{x - x_0} = f[x_0 x x]$$

$$g(x) = f[x_0, x] = \begin{cases} \dfrac{f(x) - f(x_0)}{x - x_0}, & x \neq x_0 \\ f'(x_0), & x = x_0. \end{cases}$$

Consider $\dfrac{g(x_0+h) - g(x_0)}{h} = \dfrac{f[x_0, x_0+h] - f'(x_0)}{h}$

$= \dfrac{f(x_0+h) - f(x_0) - h f'(x_0)}{h^2} = \dfrac{h^2}{2} \dfrac{f''(c)}{h^2}$, c between x_0 and x_0+h

$\longrightarrow \dfrac{f''(x_0)}{2} = f[x_0\ x_0\ x_0]$ as $h \to 0$.

Thus $g'(x_0) = \lim\limits_{h \to 0} \dfrac{g(x_0+h) - g(x_0)}{h} = f[x_0, x_0, x_0]$

Claim: $\boxed{\dfrac{d}{dx} f[x_0, x_1, x] = f[x_0, x_1, x, x]}$

Proof: Let $x_0 \neq x_1$ and $g(x) = f[x_0, x_1, x]$

Then $g(x) = f[x_0, x, x_1] = \dfrac{f[x, x_1] - f[x_0, x]}{x_1 - x_0}$

$$g'(x) = \frac{f[x_1, x, x] - f[x_0, x, x]}{x_1 - x_0}$$

$$= f[x_0, x_1, x, x]$$

Forward Difference Formula

Let $f : [c \; d] \longrightarrow \mathbb{R}$ and $x_0, x_1 \in [c, d], x_1 \neq x_0$.

$$f(x) = f(x_0) + f[x_0 \; x_1](x - x_0) + f[x_0 \; x_1 \; x]\underbrace{(x-x_0)(x-x_1)}_{\omega(x)}.$$

$$f'(x) = f[x_0, x_1] + f[x_0, x_1, x, x]\,\omega(x) + f[x_0, x_1, x]\,\omega'(x)$$

$$\omega'(x) = x - x_0 + x - x_1$$

Let $x = x_0 = a$, $x_1 = a + h$. Then $\omega(a) = 0$, $\omega'(a) = -h$

$$f'(a) \simeq f[a, a+h] = \frac{f(a+h) - f(a)}{h},$$

$$\text{error} = f[a, a+h, a](-h) = -\frac{h f''(c)}{2}$$

Central Difference Formula

$$f'(x) = f[x_0, x_1] + f[x_0, x_1, x, x]\,\omega(x) + f[x_0, x_1, x]\,\omega'(x)$$

$$\omega(x) = (x - x_0)(x - x_1), \quad \omega'(x) = x - x_0 + x - x_1$$

Let $x = a$, $x_0 = a - h$, $x_1 = a + h$. Then

$$\omega(a) = -h^2, \quad \omega'(a) = 0$$

$$f'(a) \simeq f[x_0, x_1] = \frac{f(a+h) - f(a-h)}{2h}$$

$$\text{error} = f[a-h \ \ a+h \ \ a \ \ a](-h^2) = -\frac{h^2 f^{(3)}(c)}{6}$$

It can be proved that

$$\frac{d}{dx} f[x_0, x_1, x_2, x] = f[x_0, x_1, x_2, x, x]$$

$$\frac{d}{dx} f[x_0, x_1, x_2, x, x] = 2 f[x_0, x_1, x_2, x, x, x]$$

Second derivatives

x_0, x_1, x_2 : distinct points in $[a, b]$

$p_2(x)$

$$f(x) = f(x_0) + f[x_0\ x_1](x-x_0) + f[x_0\ x_1\ x_2](x-x_0)(x-x_1)$$

$$+ f[x_0\ x_1\ x_2\ x]\underbrace{(x-x_0)(x-x_1)(x-x_2)}_{\omega(x)}$$

$$f'(x) = p_2'(x) + f[x_0, x_1, x_2, x, x]\,\omega(x)$$

$$+ f[x_0, x_1, x_2, x]\,\omega'(x)$$

$$f''(x) = p_2''(x) + 2f[x_0, x_1, x_2, x, x, x]\,\omega(x)$$

$$+ 2f[x_0, x_1, x_2, x, x]\,\omega'(x) + f[x_0, x_1, x_2, x]\,\omega''(x)$$

$$f''(x) = p_2''(x) + 2 f[x_0, x_1, x_2, x, x, x]\, \omega(x)$$

$$+ 2 f[x_0, x_1, x_2, x, x]\, \omega'(x) + f[x_0, x_1, x_2, x]\, \omega''(x)$$

$$\omega(x) = (x - x_0)(x - x_1)(x - x_2),$$

$$\omega'(x) = (x - x_0)(x - x_1) + (x - x_0)(x - x_2) + (x - x_1)(x - x_2)$$

$$\omega''(x) = 2 \{ (x - x_0) + (x - x_1) + (x - x_2) \}$$

$$x = x_0 = a \Rightarrow \omega(a) = 0,$$

$$x_1 = a + h, \quad x_2 = a + 2h \Rightarrow \omega'(a) = -2h^2, \quad \omega''(a) = -6h$$

$$x_1 = a - h, \quad x_2 = a + h \Rightarrow \omega'(a) = -h^2. \quad \omega''(a) = 0$$

$$f''(a) \simeq 2\, f[a \;\; a+h \;\; a+2h] = \frac{f(a) - 2f(a+h) + f(a+2h)}{h^2}$$

$$\text{error} = 2\, f[a \;\; a \;\; a \;\; a+h \;\; a+2h] \cdot 2h^2$$

$$+ f[a \;\; a \;\; a+h \;\; a+2h](-6h)$$

$$= \frac{f^{(4)}(c)}{6} h^2 - f'''(d)\, h$$

$$f''(a) \simeq 2\, f[a-h \;\; a \;\; a+h] = \frac{f(a-h) - 2f(a) + f(a+h)}{h^2}$$

$$\text{error} = 2\, f[a \;\; a \;\; a \;\; a-h \;\; a+h](-h^2)$$

$$= -\frac{f^{(4)}(c)}{12} h^2 \quad : \quad \underline{\text{discretization}} \;\; \underline{\text{error}}$$

Recall $\quad f'(a) = \dfrac{f(a+h) - f(a-h)}{2h} - \dfrac{h^2}{6} f'''(c)$

discretization

In calculations, we use

$f(a+h) + E_1$ and $f(a-h) + E_2$, E_1, E_2 : round-off errors.

$$f'_{comp} = \frac{f(a+h) + E_1 - (f(a-h) + E_2)}{2h}$$

$$= \frac{f(a+h) - f(a-h)}{2h} + \frac{E_1 - E_2}{2h}$$

$$f'(a) = f'_{comp} - \frac{E_1 - E_2}{2h} - \frac{h^2}{6} f'''(c)$$

does not decrease