Note Title 11/16/2010

Linear System of Equations

$$a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = b_1$$

$$a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = b_2$$

$$\vdots$$

$$a_{n1} x_1 + a_{n2} x_2 + \cdots + a_{nn} x_n = b_n$$

$$A x = b,$$

where

$A = [a_{ij}]$: $n \times n$ real matrix,

$b = [b_1, b_2, \cdots, b_n]^T$: right hand side,

$x = [x_1, x_2, \ldots, x_n]^T$: unknown vector.

$$
A_k \quad
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1k} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2k} & \cdots & a_{2n} \\
\vdots & & & & & \\
a_{k1} & a_{k2} & \cdots & a_{kk} & \cdots & a_{kn} \\
\vdots & & & & & \\
a_{n1} & a_{n2} & \cdots & a_{nk} & \cdots & a_{nn}
\end{bmatrix}
\begin{bmatrix}
x_1 \\ x_2 \\ \vdots \\ x_k \\ \vdots \\ x_n
\end{bmatrix}
=
\begin{bmatrix}
b_1 \\ b_2 \\ \vdots \\ b_k \\ \vdots \\ b_n
\end{bmatrix}
$$

Assumptions: $\det(A_k) \neq 0$, $k = 1, \cdots, n$,

where A_k : principal leading submatrix of order k.

LU decomposition

Assumption: $\det(A_k) \neq 0, \quad k = 1, 2, \cdots, n.$

A_k : principal leading submatrix of order k.

Then Gauss elimination is equivalent to writing

$$A = LU,$$

where L is unit lower triangular and

U is upper triangular matrix.

Gauss elimination

$A \longrightarrow U$,

$$L = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ m_{21} & 1 & 0 & \cdots & 0 \\ m_{31} & m_{32} & 1 & & \\ \vdots & \vdots & & 1 & \\ & & & \ddots & \\ m_{n1} & m_{n2} & & m_{n,n-1} & 1 \end{bmatrix}$$

Notations: $e_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \rightarrow$ jth place Canonical vector

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$$

$A = [a_{ij}]$: $n \times n$ matrix

$C_j = A e_j$: jth column of A $e_i^T A e_j = a_{ij}$

$R_i = e_i^T A$: ith row of A

Gauss elimination

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$m_{21} = \frac{a_{21}}{a_{11}}, \quad m_{31} = \frac{a_{31}}{a_{11}}$$

$$R_2 - m_{21} R_1, \quad R_3 - m_{31} R_1 .$$

$$a_{22}^{(1)} = a_{22} - m_{21} a_{12}, \quad a_{23}^{(1)} = a_{23} - m_{21} a_{13}$$

$$a_{32}^{(1)} = a_{32} - m_{31} a_{12}, \quad a_{33}^{(1)} = a_{33} - m_{31} a_{13}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -m_{21} & 1 & 0 \\ -m_{31} & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ 0 & a_{32}^{(1)} & a_{33}^{(1)} \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -m_{21} & 1 & 0 \\ -m_{31} & 0 & 1 \end{bmatrix} \qquad m_1 = \begin{bmatrix} 0 \\ m_{21} \\ m_{31} \end{bmatrix}$$

$$m_1 e_1^T = \begin{bmatrix} 0 \\ m_{21} \\ m_{31} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ m_{21} & 0 & 0 \\ m_{31} & 0 & 0 \end{bmatrix}$$

$$E_1 = I - m_1 e_1^T, \qquad e_1^T m_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ m_{21} \\ m_{31} \end{bmatrix} = 0$$

$$(I - m_1 e_1^T)(I + m_1 e_1^T) = I$$

First step of Gauss elimination

$$E_1 A = A^{(1)},$$

$$m_{21} = \frac{a_{21}}{a_{11}}, \quad m_{31} = \frac{a_{31}}{a_{11}},$$

$$m_1 = \begin{bmatrix} 0 \\ m_{21} \\ m_{31} \end{bmatrix}, \quad E_1 = I - m_1 e_1^T,$$

$$E_1^{-1} = I + m_1 e_1^T$$

Second Step of Gauss elimination

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ 0 & a_{32}^{(1)} & a_{33}^{(1)} \end{bmatrix}$$

$$m_{32} = \frac{a_{32}^{(1)}}{a_{22}^{(1)}}$$

$$R_3 - m_{32} R_2$$

$$a_{33}^{(2)} = a_{33}^{(1)} - m_{32}\, a_{23}^{(1)}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -m_{32} & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ 0 & a_{32}^{(1)} & a_{33}^{(1)} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ 0 & 0 & a_{33}^{(2)} \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -m_{32} & 1 \end{bmatrix}, \quad m_2 = \begin{bmatrix} 0 \\ 0 \\ m_{32} \end{bmatrix}$$

$$m_2 e_2^T = \begin{bmatrix} 0 \\ 0 \\ m_{32} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & m_{32} & 0 \end{bmatrix}, \quad e_2^T m_2 = 0$$

$$E_2 = I - m_2 e_2^T,$$

$$(I - m_2 e_2^T)(I + m_2 e_2^T) = I \implies E_2^{-1} = I + m_2 e_2^T$$

Gauss elimination

$$E_2 E_1 A = A^{(2)} = U$$

$$\Rightarrow \quad A = E_1^{-1} E_2^{-1} U$$

$$e_1^T m_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ m_{32} \end{bmatrix} = 0$$

$$E_1^{-1} E_2^{-1} = (I + m_1 e_1^T)(I + m_2 e_2^T)$$

$$= I + m_1 e_1^T + m_2 e_2^T$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{bmatrix}$$

$$m_1 = \begin{bmatrix} 0 \\ m_{21} \\ m_{31} \end{bmatrix}, \quad m_2 = \begin{bmatrix} 0 \\ 0 \\ m_{32} \end{bmatrix}$$

First step of Gauss elimination: $R_i \to R_i - m_{i1} R_1$

$$a_{ij}^{(1)} = a_{ij} - m_{i1} a_{1j}, \quad m_{i1} = \frac{a_{i1}}{a_{11}}.$$

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -m_{21} & 1 & 0 & \cdots & 0 \\ -m_{31} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & & \\ -m_{n1} & 0 & 0 & \cdots 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = A^{(1)}$$

$E_1 A = A^{(1)}$: First step of Gauss elimination

Define

$$m_1 = \begin{bmatrix} 0 \\ m_{21} \\ \vdots \\ m_{n1} \end{bmatrix}.$$

nxn matrix

$$m_1 e_1^T = [m_1 e_1^T e_1 , m_1 e_1^T e_2 , \cdots , m_1 e_1^T e_n]$$

$$= [m_1 , \bar{0} , \bar{0} , \cdots , \bar{0}]$$

$$= \begin{bmatrix} 0 & & \\ m_{21} & & \\ \vdots & & 0 \\ m_{n1} & & \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 1 & & & 0 \\ -m_{21} & 1 & & \\ \vdots & & \ddots & \\ -m_{n1} & 0 & & 1 \end{bmatrix} = I - m_1 e_1^T$$

$$E_1 = I - m_1 e_1^T$$

$$e_1^T m_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 0 \\ m_{21} \\ \vdots \\ m_{n1} \end{bmatrix} = 0$$

$$(I - m_1 e_1^T)(I + m_1 e_1^T)$$

$$= I - m_1 e_1^T + m_1 e_1^T - m_1 \underbrace{e_1^T m_1}_{=0} e_1^T = I$$

$$E_1^{-1} = I + m_1 e_1^T$$

Second step of Gauss elimination:

$$a_{ij}^{(2)} = a_{ij}^{(1)} - m_{i2}\, a_{2j}^{(1)}, \quad i, j = 3, \cdots, n.$$

Define

$$m_2 = \begin{bmatrix} 0 \\ 0 \\ m_{32} \\ \vdots \\ m_{n2} \end{bmatrix}. \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & & 0 \\ \vdots & -m_{32} & 1 & & \vdots \\ \vdots & \vdots & \vdots & & 0 \\ 0 & -m_{n2} & 0 & & 1 \end{bmatrix} = I - m_2 e_2^T$$

$$E_2 A^{(1)} = A^{(2)} \qquad E_2^{-1} = I + m_2 e_2^T$$

$$E_2 E_1 A = A^{(2)}$$

In general, for $1 \le k \le n-1$, define

$$m_k = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ m_{k+1,k} \\ \vdots \\ m_{n,k} \end{bmatrix} \quad \text{and} \quad E_k = I - m_k e_k^T .$$

$$e_1^T m_k = e_2^T m_k = \cdots = e_k^T m_k = 0$$

$$E_k^{-1} = I + m_k e_k^T$$

$$E_{n-1} E_{n-2} \cdots E_2 E_1 A = A^{(n)} = U : \text{upper triangular}$$

Thus $Ax = b$ is transformed to

$$\underbrace{E_{n-1} E_{n-2} \cdots E_2 E_1}_{= E} A x = E b,$$

that is, $Ux = y$.

Since $EA = U$ and E is an invertible matrix, both the systems have the same solution.

$$E^{-1} = E_1^{-1} E_2^{-1} \cdots E_{n-2}^{-1} E_{n-1}^{-1}$$

$$= (I + m_1 e_1^T)(I + m_2 e_2^T) \cdots (I + m_{n-1} e_{n-1}^T)$$

$$= I + m_1 e_1^T + m_2 e_2^T + \cdots + m_{n-1} e_{n-1}^T$$

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ m_{21} & 1 & & 0 \\ \vdots & m_{32} & & \vdots \\ \vdots & \vdots & & 0 \\ m_{n1} & m_{n2} & & 1 \end{bmatrix} = L$$

$$EA = U$$

$$A = E^{-1} U$$
$$= LU$$

Note that Gauss elimination can be performed provided at no stage the pivot becomes zero.

Thus $A = LU$ if at no stage the pivot becomes zero.

$$A = LU$$

$$Ax = b \iff LUx = b$$

$$\iff \quad Ly = b \quad : \text{ forward substitution}$$

$$Ux = y \quad : \text{ back substitution}$$

$$\begin{bmatrix} 1 & & & & 0 \\ l_{21} & 1 & & & \\ l_{31} & l_{32} & 1 & & \\ \vdots & & & \ddots & \\ l_{n1} & l_{n2} & \cdots & l_{n,n-1} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix} : \quad \begin{aligned} y_1 &= b_1, \\ y_2 &= b_2 - l_{21} y_1, \\ y_i &= b_i - \sum_{j=1}^{i-1} l_{ij} y_j \end{aligned}$$

$Ux = y$: as before

Stoarage :

$$A \longrightarrow \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ l_{21} & u_{22} & u_{23} & \cdots & u_{2n} \\ l_{31} & l_{32} & u_{33} & \cdots & u_{3n} \\ \vdots & & & & \\ l_{n1} & l_{n2} & \cdots & l_{n,n-1} & u_{nn} \end{bmatrix} : l_{ii} = 1, \quad i = 1, \cdots, n$$

LU decomposition : Uniqueness

A : $n \times n$ matrix, leading principal submatrices : nonsingular.

Then $A = LU$, where

L : unit lower triangular, U : upper triangular, invertible.

Let $A = L_1 U_1 = L_2 U_2$.

Then $\underbrace{L_2^{-1} L_1}_{\text{unit lower triangular}} = \underbrace{U_2 U_1^{-1}}_{\text{upper triangular}} = I$

$\implies L_1 = L_2, \quad U_1 = U_2.$

Suppose we know that A can be written as LU

Then we can determine L and U directly as follows.

$$A = \begin{bmatrix} 1 & & & & & \\ l_{21} & 1 & & & 0 & \\ l_{31} & l_{32} & 1 & & & \\ \vdots & & & & & \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{n,n-1} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & & u_{1n} \\ 0 & u_{22} & u_{23} & & & u_{2n} \\ 0 & 0 & u_{33} & & & u_{3n} \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & 0 & u_{nn} \end{bmatrix}$$

$\Rightarrow a_{1j} = u_{1j}, \; j = 1, \cdots, n$ (1st row of U is determined)

$l_{i1} u_{11} = a_{i1}, \; i = 2, \cdots, n$

$\Rightarrow l_{i1} = \dfrac{a_{i1}}{u_{11}}$ (1st column of L is determined) known

known

$$A = \begin{bmatrix} 1 & & & & & \\ l_{21} & 1 & & & 0 & \\ l_{31} & l_{32} & 1 & & & \\ \vdots & & & & & \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{n,n-1} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & & u_{1n} \\ 0 & u_{22} & u_{23} & & & u_{2n} \\ 0 & 0 & u_{33} & & & u_{3n} \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & 0 & u_{nn} \end{bmatrix}$$

2nd row of L x jth column of U $(j = 2, \cdots, n)$

$$a_{2j} = l_{21} u_{1j} + u_{2j}, \quad j = 2, \cdots, n$$

(2nd row of U is determined)
(also 2nd Column of U)

ith row of L x 2nd Column of U

$$a_{i2} = l_{i1} u_{12} + l_{i2} u_{22}, \quad i = 3, \cdots, n$$

2nd column (and row) of L is determined.

LDV decomposition

$A = LU$, where $u_{ii} \neq 0$.

Define $D = \text{diag.}(u_{11}\ u_{22} \cdots u_{nn})$ and

$$V = D^{-1} U.$$

$$v_{ii} = \sum_{k=1}^{n} D^{-1}(i,k)\, U(k,i) = D^{-1}(i,i)\, u_{ii} = 1 \,:$$

V : unit upper triangular, $U = DV$ and

$$A = L\,D\,V$$

$A = LDV$, L : unit lower triangular,
D : diagonal,
V : unit upper triangular.

uniqueness of the decomposition :

$$A = L_1 D_1 V_1 = L_2 D_2 V_2$$

$\Rightarrow L_1 = L_2$, $D_1 V_1 = D_2 V_2$ (uniqueness of the LV decomposition)

$$\Rightarrow \underbrace{D_2^{-1} D_1}_{\text{diagonal}} = \underbrace{V_2 V_1^{-1}}_{\text{unit upper triangular}} = I \quad \Rightarrow D_1 = D_2 , V_1 = V_2$$