Note Title 3/24/2011

LDV decomposition

$A = [a_{ij}]$: $n \times n$ matrix, $\det(A_k) \neq 0$, $k = 1, \cdots, n$

$A = LU$, L : unit lower triangular,

U : upper triangular

$\det(A) = \det(L)\det(U) = u_{11} u_{22} \cdots u_{nn} \neq 0$,

$u_{ii} \neq 0$, $i = 1, \cdots, n$

$D = \mathrm{diag}(u_{11}, u_{22}, \cdots, u_{nn})$, $V = D^{-1} U$

V : unit upper triangular

$A = LU$, $\quad V = D^{-1} U$

$\Rightarrow A = LDV$: unique

L : unit lower triangular

D : diagonal , U : unit upper triangular

A : symmetric : $A^T = A$

$\det(A_k) \neq 0$, $k = 1, \cdots, n$

$A = LDV \implies A^T = V^T D L^T$

$$\begin{aligned} A^T &= V^T D L^T \\ \| \\ A &= L D V \end{aligned}$$

$\implies V = L^T$

$$A = LDL^T$$

A : positive definite, i.e.,

$A^T = A$, $x \neq \bar{0} \Rightarrow x^T A x > 0$

$\det(A_k) > 0$, $k = 1, \cdots, n$

$A = L D L^T \Rightarrow D = L^{-1} A (L^T)^{-1}$.

$d_{ii} = e_i^T D e_i = e_i^T L^{-1} A (L^T)^{-1} e_i$

Let $y = (L^T)^{-1} e_i \neq 0$, $y^T = e_i^T L^{-1}$.

$d_{ii} = y^T A y > 0$

Cholesky decomposition

A : positive definite

$$A = L D L^T, \quad D = \text{diag}\cdot(d_{11}, \cdots, d_{nn}),$$
$$d_{ii} > 0.$$

Define $D^{1/2} = \text{diag}\cdot(\sqrt{d_{11}}, \cdots, \sqrt{d_{nn}})$.

Then $(D^{1/2})^2 = D$.

$$A = L D^{1/2} D^{1/2} L^T = G G^T$$

G : lower triangular matrix

Consider $A = M M^T$. Then $A^T = A$.

Let $x \neq \overline{0}$. If M is invertible, then $Mx \neq \overline{0}$.

Hence

$$x^T A x = x^T M M^T x = (M^T x)^T M^T x > 0$$

and

A is <u>positive</u>-<u>definite</u>.

A : $n \times n$ posite-definite matrix $\therefore$ $A = GG^T$

$$\begin{bmatrix} g_{11} & & & & 0 \\ g_{21} & g_{22} & & & \\ g_{31} & g_{32} & g_{33} & & \\ \vdots & & & & \\ g_{n1} & g_{n2} & g_{n3} & \cdots & g_{nn} \end{bmatrix} \begin{bmatrix} g_{11} & g_{21} & g_{31} & \cdots & g_{n1} \\ 0 & g_{22} & g_{32} & \cdots & g_{n2} \\ 0 & 0 & g_{33} & \cdots & g_{n3} \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & g_{nn} \end{bmatrix} = A.$$

As in LU decomposition,

1st step : determine 1st row of G^T (= 1st column of G).

2nd step : determine 2nd row of G^T (= 2nd column of G)

1st row of G x jth Column of G^T

$$A = G G^T : \quad a_{1j} = \sum_{k=1}^{n} G(1,k) G^T(k,j) = g_{11} g_{j1},$$

$$j = 1, \cdots, n$$

$$g_{11}^2 = a_{11}, \quad a_{j1} = a_{1j} = g_{11} g_{j1}, \; j = 2, \cdots, n$$

$$g_{11} = \sqrt{a_{11}}, \quad g_{j1} = \frac{a_{j1}}{g_{11}}, \; j = 2, \cdots, n .$$

2nd row of G x jth column of G^T : $j = 2, \cdots, n$

$$g_{21}^2 + g_{22}^2 = a_{22}, \quad g_{21} g_{j1} + g_{22} g_{j2} = a_{j2}, \; j = 3, \cdots, n$$

$$g_{22} = \sqrt{a_{22} - g_{21}^2}, \quad g_{j2} = \frac{a_{j2} - g_{22} g_{j2}}{g_{22}}, \; j = 3, \cdots, n .$$

At each stage of deciding the diagonal entries g_{ii} of G, we can choose positive or negative square root. If we decide to choose $g_{ii} > 0$, we obtain

$$A = GG^T$$

and it proves the uniqueness of the decomposition.

Number of operations $= O\left(\frac{n^3}{6}\right)$

Recall that : A is positive-definite if

$$A^T = A \,, \quad x^T A x > 0 \quad \text{for } x \neq \bar{0}$$

difficult to verify.

Try writing $A = G G^T$. If the algorithm fails (number under square-root is zero or negative), then A is not positive-definite.

Practical test.

Note Title 11/27/2010

Gauss elimination with partial pivoting

$$Ax = b$$

Assumption : A is invertible ; $\det(A) \neq 0$.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

<u>Assumption</u> : A is invertible.

It is possible that $a_{11} = 0$.

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$: invertible.

Even if $a_{11} \neq 0$, if we apply Gauss elimination process, it is possible that subsequently the pivot becomes zero.

<u>Assumption</u>: A is invertible.

It is possible that $a_{11} \neq 0$, but it is small.

The division by a small number should be avoided (discussion later).

We consider the elements in the first column and let $|a_{k1}| = \max\limits_{1 \le i \le n} |a_{i1}|$.

Then $a_{k1} \neq 0$ (why?)

Interchange the first and the k th row.

Interchange the first and the k-th row :

$$\begin{array}{c} \rightarrow \\ \\ \\ \rightarrow \\ \\ \\ \end{array} \begin{bmatrix} a_{k1} & a_{k2} & \cdots & a_{kn} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_k \\ b_2 \\ \vdots \\ b_1 \\ \vdots \\ b_n \end{bmatrix}$$

Define $m_{i1} = \dfrac{a_{i1}}{a_{k1}}$, $i \neq k$, $i = 1, \cdots, n$

$$\tilde{R}_i \longrightarrow \tilde{R}_i - m_{i1} \tilde{R}_1$$

Note that

$|m_{i1}| \leq 1$

$$
\begin{bmatrix}
\tilde{a}_{11} & \tilde{a}_{21} & \cdots & \tilde{a}_{n1} \\
0 & \tilde{a}_{22}^{(1)} & \cdots & \tilde{a}_{2n}^{(1)} \\
\vdots & & & \\
0 & \tilde{a}_{n2}^{(1)} & \cdots & \tilde{a}_{nn}^{(1)}
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}
=
\begin{bmatrix} \tilde{b}_1 \\ \tilde{b}_2^{(1)} \\ \vdots \\ \tilde{b}_n^{(1)} \end{bmatrix}
$$

Let $\quad |\tilde{a}_{k2}^{(1)}| = \max\limits_{2 \le i \le n} |\tilde{a}_{i2}^{(1)}| \neq 0 \quad$ (why ?)

Interchange 2nd and kth equation, define

$$m_{i2} = \frac{\tilde{a}_{i2}^{(1)}}{\tilde{a}_{k2}^{(1)}}, \quad i \neq k, \quad i = 2, \cdots, n$$ and perform

Continue ...

$$\bar{R}_i \longrightarrow \bar{R}_i - m_{i2} \bar{R}_2, \quad i = 3, \cdots, n$$