Note Title 2/29/2012

Approximation by Polynomials

$C[a,b] = \{ f : [a,b] \longrightarrow \mathbb{R} \text{ continuous} \}$

vector space

$\|f\|_\infty = \max \{ |f(x)| : x \in [a,b] \}$

Polynomials

$$p_n(x) = a_0 + a_1 x + \cdots + a_n x^n,$$ Power form

$$a_0, \ldots, a_n \in \mathbb{R}, \quad x \in \mathbb{R}$$

$a_n \neq 0$: p_n : polynomial of degree n

$a_n = 0$: p_n : polynomial of degree $< n$

$$p_n'(x) = a_1 + 2a_2 x + \cdots + n a_n x^{n-1}$$

$$\int p_n(x)\,dx = a_0 x + a_1 \frac{x^2}{2} + \cdots + a_n \frac{x^{n+1}}{n+1} + C$$

Approximation of functions by polynomials

$f : [a,b] \to \mathbb{R}$, $c \in [a,b]$,

$f, f', \dots, f^{(n)} \in C[a,b]$,

$f^{(n)}$: differentiable on (a,b)

$$f(x) = f(c) + f'(c)(x-c) + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \frac{f^{(n+1)}(d_x)}{(n+1)!}(x-c)^{n+1}$$

$f(x) \simeq p_n(x)$ in a neighbourhood of c.

Weierstrass Theorem

$f : [a, b] \longrightarrow \mathbb{R}$ continuous

Then there exists a sequence of polynomials $p_n(x)$ such that

$$\| f - p_n \|_\infty = \max_{x \in [a,b]} | f(x) - p_n(x) | \longrightarrow 0$$

as $n \longrightarrow \infty$.

$f : [0,1] \longrightarrow \mathbb{R}$ continuous

$$B_n(f)(x) = \sum_{i=0}^{n} \frac{n!}{k!\,(n-k)!} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}$$

Bernstein Polynomial

$B_n(f)$: polynomial of degree $\leq n$

$\| f - B_n(f) \|_\infty \longrightarrow 0$ as $n \to \infty$

Bernstein polynomials of special functions

$$B_n(f)(x) = \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}$$

$f(x) \equiv 1$

$$\Rightarrow B_n(f)(x) = \sum_{k=0}^{n} {}^nC_k\, x^k (1-x)^{n-k}$$

$$= (x + 1 - x)^n = 1 = f(x)$$

$\| B_n(f) - f \|_\infty = 0 \quad , \quad n \geq 0$

$$B_n(f)(x) = \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}$$

$f(x) = x$

$$\Rightarrow B_n(f)(x) = \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} \frac{k}{n} x^k (1-x)^{n-k}$$

$$= x \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!\,(n-1-(k-1))!} x^{k-1} (1-x)^{n-1-(k-1)}$$

$$= x\,(x+1-x)^{n-1} = x \quad \text{for } n \geq 1$$

$B_0(f)(x) = 0$

$$B_n(f)(x) = \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}$$

$$f(x) = x^2$$

$$\Rightarrow B_n(f)(x) = \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} \left(\frac{k}{n}\right)^2 x^k (1-x)^{n-k}$$

$$= \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!\,(n-k)!} \frac{k-1+1}{n} x^k (1-x)^{n-k}$$

$$= \frac{(n-1)}{n} x^2 \underbrace{\sum_{k=2}^{n} \frac{(n-2)!}{(k-2)!\,(n-k)!} x^k (1-x)^{n-k}}_{=1} + \frac{x}{n}$$

$$f(x) = x^2$$

$$B_n(f)(x) = \frac{n-1}{n} x^2 + \frac{1}{n} x \;, \quad n \geqslant 2$$

$$= x^2 + \frac{1}{n} x(1-x) \;, \quad B_0(f)(x) = 0 \;, \quad B_1(f)(x) = 1$$

$$\| B_n f - f \|_\infty = \max_{x \in [0,1]} \left| \frac{1}{n} x(1-x) \right|$$

$$= \frac{1}{4n} \longrightarrow 0 \quad \text{as} \quad n \to \infty$$

$f(x) = 1 \implies B_n(f)(x) = 1, \quad n \geq 0$

$\| f - B_n(f) \|_\infty = 0$ for all n

$f(x) = x \implies B_n(f)(x) = x, \quad n \geq 1$

$B_0(f)(x) = 0$

$\| f - B_n(f) \|_\infty = 0$ for $n \geq 1$

$f(x) = x^2 \implies B_n(f)(x) = x^2 + \frac{1}{n} x(1-x), \quad n \geq 2$

$B_0(f)(x) = 0, \quad B_1(f)(x) = x$

For $n \geq 2$,

$\| f - B_n(f) \|_\infty = \frac{1}{4n} \to 0$ as $n \to \infty$

$$B_n(f)(x) = \sum_{k=0}^{n} {}^nC_k \, f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}$$

$$B_n(f+g)(x) = \sum_{k=0}^{n} {}^nC_k \, (f+g)\left(\frac{k}{n}\right) x^k (1-x)^{n-k}$$

$$= \sum_{k=0}^{n} {}^nC_k \, f\left(\frac{k}{n}\right) x^k (1-x)^{n-k} + \sum_{k=0}^{n} {}^nC_k \, g\left(\frac{k}{n}\right) x^k (1-x)^{n-k}$$

$$= B_n(f)(x) + B_n(g)(x)$$

$$B_n(\alpha f)(x) = \alpha \, B_n(f)(x) \quad , \alpha \in \mathbb{R}$$

$$B_n(f)(x) = \sum_{k=0}^{n} {}^nC_k \, f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}$$

$x \in [0,1]$, $\displaystyle\sum_{k=0}^{n} {}^nC_k \, x^k (1-x)^{n-k}$

$$= (x + 1 - x)^n = 1$$

Fix $x = x_0 \in [0,1]$

$$B_n(f)(x_0) = \sum_{k=0}^{n} {}^nC_k \, x_0^k (1-x_0)^{n-k} f\left(\frac{k}{n}\right)$$

$$= \sum_{k=0}^{n} \alpha_k \, f\left(\frac{k}{n}\right), \quad \alpha_k \geq 0, \ \sum_{k=0}^{n} \alpha_k = 1$$

$$B_n(f)(x_0) = \sum_{k=0}^{n} \alpha_k f\left(\frac{k}{n}\right),$$

$$\alpha_k \geq 0, \quad \sum_{k=0}^{n} \alpha_k = 1.$$

$$f(x) \geq 0, \quad x \in [0,1] \implies f\left(\frac{k}{n}\right) \geq 0,$$
$$k = 0, 1, \cdots, n$$

$\implies B_n(f)(x_0) \geq 0$ for each $x_0 \in [0,1]$

$\implies B_n(f) \geq 0$

$$B_n : C[0,1] \longrightarrow C[0,1]$$

$$B_n(f)(x) = \sum_{k=0}^{n} {}^nC_k \, f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}$$

$$B_n(\alpha f + g) = \alpha B_n(f) + B_n(g),$$

$\alpha \in \mathbb{R}$, $f, g \in [0,1]$ linear

$f \geqslant 0 \implies B_n(f) \geqslant 0$: positive

Korovkin Theorem

Let $P_n : C[0,1] \longrightarrow C[0,1]$ be a positive linear map. If

$\| P_n(f) - f \|_\infty \longrightarrow 0$ for $f(x) = 1, x, x^2,$

then $\| P_n(f) - f \|_\infty \longrightarrow 0$

for each $f \in C[0,1]$.

$f(x) = x^2$, $\| B_n(f) - f \|_\infty = \frac{1}{4n}$

$\| B_n(f) - f \|_\infty < 10^{-6}$

$\iff \frac{1}{4n} < 10^{-6} \iff 25 \times 10^4 < n$

$n > 250000$

Slow Convergence,

f polynomial $\not\Rightarrow$ $f = B_n(f)$ for all n large enough

Best Approximation

There exists a unique polynomial p_n^* of degree $\le n$ such that

$$\| f - p_n^* \|_\infty \le \min_{\substack{p_n :\ \text{poly. of} \\ \text{degree} \le n}} \| f - p_n \|_\infty$$

p_n^* : Computation needs iteration technique called the second algorithm of Remes

Interpolating Polynomials

$f : [a, b] \longrightarrow \mathbb{R}$

$x_0, x_1, \cdots, x_n$: distinct points in $[a, b]$

Then there exists a unique polynomial p_n of degree $\leq n$ such that

$$p_n(x_j) = f(x_j), \; j = 0, 1, \cdots, n$$

Interpolation Points

$x_{0,0}$ $\qquad$ p_0

$x_{1,0}\,,\ x_{1,1}$ $\qquad$ p_1

$x_{2,0}\,,\ x_{2,1}\,,\ x_{2,2}$ $\qquad$ p_2

$\vdots$

$x_{n,0}\,,\ x_{n,1}\,,\ \cdots\ x_{n,n}$ $\qquad$ p_n

Question: For each $f \in C[a,b]$, does $\|f - p_n\|_\infty \longrightarrow 0$ as $n \to \infty$?

Interpolating Polynomials : No Convergence

High degree polynomials : Stability Problem

$$p(x) = (x-1)(x-2)(x-3)(x-4)(x-5)(x-6)(x-7)$$
$$= x^7 - 28x^6 + 322x^5 - - -$$

If we change -28 to -28.002,

the original roots are perturbed to

$-5.459 \pm 0.540\, i$

low degree polynomial interpolation with an appropriate choice of interpolation points.

Piecewise polynomials