Note Title 1/21/2011

Vector norm

$\|\cdot\| : \mathbb{R}^n \longrightarrow \mathbb{R}^+$ such that

1. $\|x\| \geq 0$, $\|x\| = 0 \Rightarrow x = \bar{0}$

2. $\|\alpha x\| = |\alpha| \, \|x\|$, $\alpha \in \mathbb{R}$, $x \in \mathbb{R}^n$

3. $\|x + y\| \leq \|x\| + \|y\|$, $x, y \in \mathbb{R}^n$

Examples : $x = [x_1 \; x_2 \cdots x_n]^T$

$$\|x\|_2 = \left(\sum_{j=1}^{n} x_j^2 \right)^{1/2} \quad : \quad \text{Euclidean norm}$$

$$\|x\|_1 = \sum_{j=1}^{n} |x_j| \quad : \quad 1-\text{norm}$$

$$\|x\|_\infty = \max_{1 \le j \le n} |x_j| \quad : \quad \infty-\text{norm}$$

Inner Product on $\mathbb{R}^n$

$$\langle x, y \rangle = \sum_{j=1}^{n} x_j \, y_j$$

$$\langle x, x \rangle^{1/2} = \left(\sum_{j=1}^{n} x_j^2 \right)^{1/2} = \|x\|_2$$

Cauchy-Schwarz Inequality:

$$| \langle x, y \rangle | \leq \|x\|_2 \, \|y\|_2$$

$$\left| \sum_{j=1}^{n} x_j \, y_j \right| \leq \left(\sum_{j=1}^{n} x_j^2 \right)^{1/2} \left(\sum_{j=1}^{n} y_j^2 \right)^{1/2}$$

Cauchy-Schwarz Inequality

$$\left|\sum_{j=1}^{n} x_j y_j\right| \leq \left(\sum_{j=1}^{n} x_j^2\right)^{1/2} \left(\sum_{j=1}^{n} y_j^2\right)^{1/2}$$

$$\left(\|x+y\|_2\right)^2 = \sum_{j=1}^{n} (x_j + y_j)^2$$

$$= \sum_{j=1}^{n} x_j^2 + \sum_{j=1}^{n} y_j^2 + 2 \sum_{j=1}^{n} x_j y_j$$

$$\leq \left(\|x\|_2\right)^2 + \left(\|y\|_2\right)^2 + 2\|x\|_2 \|y\|_2 = \left(\|x\|_2 + \|y\|_2\right)^2$$

$A = [a_{ij}]$ $n \times n$ matrix.

$$\|A\|_F = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 \right)^{1/2}$$: <u>Frobenius</u> <u>norm</u>.

$$\|A\|_{max} = \max_{\substack{1 \le i \le n \\ 1 \le j \le n}} |a_{ij}|$$

Induced Matrix Norm.

A : $n \times n$ matrix. Fix a vector norm. Define

$$\|A\| = \max_{x \neq \bar{0}} \frac{\|Ax\|}{\|x\|}$$

1. $\|A\| \geq 0$, $\|A\| = 0 \Rightarrow \|Ax\| = 0$ for $\bar{0} \neq x \in \mathbb{R}^n$

$\Rightarrow \|A e_j\| = 0$, $j = 1, \cdots, n \Rightarrow C_j = A e_j = \bar{0}$

$\Rightarrow A$: zero matrix.

2. $$\|\alpha A\| = \max_{x \neq \bar{0}} \frac{\|(\alpha A)x\|}{\|x\|} = |\alpha| \, \|A\|$$

3. $$\|A+B\| = \max_{x \neq \bar{0}} \frac{\|(A+B)x\|}{\|x\|} \leq \|A\| + \|B\|$$

$$\|A\| = \max_{x \neq \bar{0}} \frac{\|Ax\|}{\|x\|}$$

$$\Rightarrow \frac{\|Ax\|}{\|x\|} \leq \|A\| \quad \text{for all } \bar{0} \neq x \in \mathbb{R}^n$$

$$\Rightarrow \boxed{\|Ax\| \leq \|A\| \, \|x\|} \quad \text{for all } x \in \mathbb{R}^n .$$

Consider $\|ABx\| \leq \|A\| \, \|Bx\| \leq \|A\| \, \|B\| \, \|x\|$

$\Rightarrow \dfrac{\|ABx\|}{\|x\|} \leq \|A\| \, \|B\|$ for $\bar{0} \neq x \in \mathbb{R}^n$

$\Rightarrow \|AB\| \leq \|A\| \, \|B\|$: Consistency condition

Formula for calculating $\|A\|_1$

$$(Ax)(i) = \sum_{j=1}^{n} a_{ij} x_j \qquad \|Ax\|_1 = \sum_{i=1}^{n} \left| \sum_{j=1}^{n} a_{ij} x_j \right|$$

$$\|Ax\|_1 \leq \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}| \, |x_j| = \sum_{j=1}^{n} |x_j| \left(\sum_{i=1}^{n} |a_{ij}| \right)$$

$$\leq \left(\max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}| \right) \sum_{j=1}^{n} |x_j|$$

$$= \alpha \, \|x\|_1$$

$$\| Ax \|_1 \leq \alpha \, \| x \|_1 , \quad \text{where}$$

$$\alpha = \max_{1 \leq j \leq n} \sum_{i=1}^{n} | a_{ij} |$$

For $x \neq \bar{0}$, $\dfrac{\| Ax \|_1}{\| x \|_1} \leq \alpha$

$$\Rightarrow \| A \|_1 = \max_{x \neq \bar{0}} \frac{\| Ax \|_1}{\| x \|_1} \leq \alpha .$$

$$\|A\|_1 \leq \alpha = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}| = \sum_{i=1}^{n} |a_{ij_0}|$$

for some j_0

Consider $A e_{j_0} = \begin{bmatrix} a_{1j_0} \\ a_{2j_0} \\ \vdots \\ a_{nj_0} \end{bmatrix}$.

$\|A e_{j_0}\|_1 = \alpha$,

$\|e_{j_0}\|_1 = 1$.

$$\alpha = \frac{\|A e_{j_0}\|_1}{\|e_{j_0}\|_1} \leq \|A\|_1 .$$

$\|A\|_1 = \alpha$

Column-sum norm

∞ - norm of a matrix

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$$\|Ax\|_\infty = \max_{1 \le i \le n} |(Ax)(i)|$$

$$= \max_{1 \le i \le n} \left| \sum_{j=1}^{n} a_{ij} x_j \right|$$

$$\le \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}| |x_j|$$

$$\le \|x\|_\infty \underbrace{\max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|}_{= \beta}$$

$$\le \beta \|x\|_\infty$$

$$\|A\|_\infty \le \beta$$

$$\|Ax\|_\infty \le \beta \|x\|_\infty \qquad \beta = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}| = \sum_{j=1}^{n} |a_{i_0 j}|$$

Define $y_j = \begin{cases} \dfrac{|a_{i_0,j}|}{a_{i_0,j}}, & a_{i_0,j} \neq 0 \\ 0, & a_{i_0,j} = 0 \end{cases}$ $\qquad \|y\|_\infty = 1.$

$$(Ay)(i_0) = \sum_{j=1}^{n} a_{i_0,j}\, y_j = \sum_{j=1}^{n} |a_{i_0,j}| = \beta$$

$$\beta = |Ay(i_0)| \le \|Ay\|_\infty \le \|A\|_\infty \|y\|_\infty = \|A\|_\infty$$

2-norm of the matrix (upper bound)

$$\|A\|_2 = \max_{x \neq \bar{0}} \frac{\|Ax\|_2}{\|x\|_2}, \quad \|x\|_2 = \left(\sum_{j=1}^{n} x_j^2\right)^{1/2}$$

$$\|Ax\|_2^2 = \sum_{i=1}^{n} Ax(i)^2 = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} x_j\right)^2$$

$$\leq \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij}^2\right)\left(\sum_{j=1}^{n} x_j^2\right)$$ (using Cauchy-Schwarz inequality)

$$\leq \left(\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2\right) \|x\|_2^2 = \|A\|_F^2 \, \|x\|_2^2$$

$$\|Ax\|_2 \leq \|A\|_F \, \|x\|_2$$

For $\bar{0} \neq x \in \mathbb{R}^n$, $\dfrac{\|Ax\|_2}{\|x\|_2} \leq \|A\|_F$

$$\Rightarrow \quad \|A\|_2 = \max_{x \neq \bar{0}} \frac{\|Ax\|_2}{\|x\|_2}$$

$$\leq \|A\|_F = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 \right)^{1/2}$$

Matrix Norms

$$\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}| \quad : \quad \text{Column-sum norm}$$

$$\|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}| \quad : \quad \text{Row-sum norm}$$

$$\|A\|_F = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 \right)^{1/2} \quad : \quad \text{Frobenius norm}$$

$$\|A\|_2 \le \|A\|_F$$