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# Sensitivity of Linear Systems.

Consider $Ax = b$,

where $A$ is invertible and $b \neq 0$.

In practice, we solve

$$(A + \delta A)(\hat{x}) = b + \delta b.$$

Aim: To bound $\dfrac{\|x - \hat{x}\|}{\|x\|}$

..

Exact System : $Ax = b$.

Perturbed System : $A\hat{x} = b + \delta b$,

$\delta b$ : 'small'.

Let $\hat{x} = x + \delta x$. $(\delta x = \hat{x} - x)$

Claim: $$\frac{\|\delta x\|}{\|x\|} \leq \left(\|A\| \, \|A^{-1}\|\right) \frac{\|\delta b\|}{\|b\|}$$

$$Ax = b, \quad A(x+\delta x) = b + \delta b \Rightarrow A\,\delta x = \delta b$$

$$\|b\| = \|Ax\| \le \|A\|\,\|x\|$$

$$\frac{1}{\|x\|} \le \frac{\|A\|}{\|b\|} \quad \cdots (1)$$

$$\delta x = A^{-1}\delta b \Rightarrow \|\delta x\| \le \|A^{-1}\|\,\|\delta b\| \quad \cdots (2)$$

From (1) and (2), $\dfrac{\|\delta x\|}{\|x\|} \le \left(\|A\|\,\|A^{-1}\|\right)\dfrac{\|\delta b\|}{\|b\|}$

$$\frac{\|\delta x\|}{\|x\|} \leq \|A\| \, \|A^{-1}\| \, \frac{\|\delta b\|}{\|b\|}$$

$k(A) = \|A\| \, \|A^{-1}\|$ : <u>Condition</u> <u>number</u>

If $k(A)$ is small, then

$\frac{\|\delta b\|}{\|b\|}$ small $\Rightarrow$ $\frac{\|\delta x\|}{\|x\|}$ small, i.e.,

the system is not overly sensitive to perturbations in $b$.

$$k(A) = \|A\| \, \|A^{-1}\| ,$$

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|} \implies \|I\| = \max_{x \neq \bar{0}} \frac{\|x\|}{\|x\|} = 1$$

$$1 = \|I\| = \|A A^{-1}\| \leq \|A\| \, \|A^{-1}\| = k(A)$$

Best possible condition number : $k(A) = 1$ .

$\mathbb{R}^n$ $\quad \langle x, y \rangle = y^t x = \sum_{j=1}^{n} x_j y_j$

$$\|x\|_2 = (x^t x)^{1/2} = \left(\sum_{j=1}^{n} x_j^2\right)^{1/2}$$

Claim: $Q$ : $n \times n$ orthogonal matrix, i.e.,

$$Q^t Q = I$$

$$\Rightarrow \kappa_2(Q) = \|Q\|_2 \|Q^{-1}\|_2 = 1$$

$Q$ : orthogonal matrix : $Q^t Q = I = Q Q^t$

$$\|Qx\|_2^2 = \langle Qx, Qx \rangle = (Qx)^t Qx$$
$$= x^t Q^t Q x = x^t x = \|x\|_2^2$$

For $x \neq \bar{0}$, $\dfrac{\|Qx\|_2}{\|x\|_2} = 1 \implies \|Q\|_2 = 1$

$(Q^t)^t Q^t = Q Q^t = I \implies Q^t$ : orthogonal

$\|Q^t\| = \|Q^{-1}\|_2 = 1 \implies k_2(Q) = 1$

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}, \qquad Q^t = Q, \quad Q^2 = I.$$

$$\|Q\|_2 = 1.$$

$$\|Q\|_1 = \|Q\|_\infty = \sqrt{2}.$$

P : permutation matrix.

obtained from the Identity matrix by interchange of two rows.

$P^T = P$ , $P^2 = I$

$\| P \|_2 = 1 \Rightarrow k_2(P) = 1$

$\| P \|_1 = \| P \|_\infty = 1 \Rightarrow k_1(P) = k_\infty(P) = 1$

$P_1, P_2$ : permutation matrices :

$$(P_1 P_2)(P_1 P_2)^T = P_1 P_2 P_2^T P_1^T = P_1 P_1^T = I$$

$$\Rightarrow k_2(P_1 P_2) = 1 .$$

Also, $k_1(P_1 P_2) = k_\infty(P_1 P_2) = 1$

## Lower Bound for $\frac{\|\delta x\|}{\|x\|}$

$$Ax = b \implies x = A^{-1}b \implies \|x\| \le \|A^{-1}\| \, \|b\|$$

$$\implies \frac{1}{\|A^{-1}\| \, \|b\|} \le \frac{1}{\|x\|}$$

$$A\,\delta x = \delta b \implies \|\delta b\| \le \|A\| \, \|\delta x\|$$

$$\implies \frac{\|\delta b\|}{\|A\|} \le \|\delta x\|$$

$$\frac{1}{\|A\| \, \|A^{-1}\|} \frac{\|\delta b\|}{\|b\|} \le \frac{\|\delta x\|}{\|x\|} \le \left(\|A\| \, \|A^{-1}\|\right) \frac{\|\delta b\|}{\|b\|}$$

$$k(A) = 1 \implies \frac{\|\delta x\|}{\|x\|} = \frac{\|\delta b\|}{\|b\|}$$

# Ill-Conditioned Matrix

$$A = \begin{bmatrix} 1000 & 999 \\ 999 & 998 \end{bmatrix}, \quad \|A\|_1 = \|A\|_\infty = 1999$$

$$A^{-1} = \begin{bmatrix} -998 & 999 \\ 999 & -1000 \end{bmatrix}, \quad \|A^{-1}\|_1 = \|A^{-1}\|_\infty = 1999$$

$$k_\infty(A) = k_1(A) = (1999)^2 \simeq 3.996 \times 10^6$$

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Consider
$$\begin{bmatrix} 1000 & 999 \\ 999 & 998 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1999 \\ 1997 \end{bmatrix}$$

Perturbed System:
$$\begin{bmatrix} 1000 & 999 \\ 999 & 998 \end{bmatrix}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 1998.99 \\ 1997.01 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 20.97 \\ -18.99 \end{bmatrix}$$

$$\frac{\|\delta b\|_\infty}{\|b\|_\infty} = \frac{0.01}{1999} \simeq 5.0 \times 10^{-6}, \quad \frac{\|\delta x\|_\infty}{\|x\|_\infty} = 19.99$$

Let $C$ be an $n \times n$ matrix such that $\|C\| < 1$.

Consider $(I + C)x = \bar{0} \implies x = -Cx$

$\implies \|x\| = \|Cx\| \leq \|C\| \, \|x\|$

$\implies \|x\| = \bar{0} \implies x = \bar{0}$.

Thus $I + C$ is 1-1 and hence invertible.

Let $A$ be invertible and $\delta A$ be such that

$$\frac{\|\delta A\|}{\|A\|} < \frac{1}{k(A)} = \frac{1}{\|A\| \, \|A^{-1}\|} \Rightarrow \|\delta A\| \, \|A^{-1}\| < 1$$

$A + \delta A = (I + \delta A \, A^{-1}) A$ is then invertible.

# Perturbation of A

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$Ax = b$ , $(A + \delta A)\hat{x} = b$ , $A$ : invertible.

$$\frac{\|\delta A\|}{\|A\|} < \frac{1}{k(A)} \Rightarrow A + \delta A \text{ is invertible}.$$

$$Ax = b, \qquad (A + \delta A)(x + \delta x) = b$$

$$\Rightarrow \quad A\,\delta x = -\delta A\,(x + \delta x)$$

$$\Rightarrow \quad \|\delta x\| \le \|A^{-1}\|\,\|\delta A\|\,(\|x\| + \|\delta x\|)$$

$$= k(A)\,\frac{\|\delta A\|}{\|A\|}\,(\|x\| + \|\delta x\|)$$

$$\Rightarrow \quad \left(1 - k(A)\,\frac{\|\delta A\|}{\|A\|}\right)\|\delta x\| \le k(A)\,\frac{\|\delta A\|}{\|A\|}\,\|x\|$$

$$\frac{\|\delta x\|}{\|x\|} \le \left(\frac{k(A)}{1 - k(A)\,\frac{\|\delta A\|}{\|A\|}}\right)\frac{\|\delta A\|}{\|A\|}$$