Note Title 4/1/2011

Perturbation

$$Ax = b, \qquad A(x+\delta x) = b + \delta b$$

$$\frac{\|\delta x\|}{\|x\|} \le k(A) \frac{\|\delta A\|}{\|A\|}, \qquad k(A) = \|A\|\,\|A^{-1}\|$$

$$Ax = b, \qquad (A+\delta A)(x+\delta x) = b \qquad \boxed{\frac{\|\delta A\|}{\|A\|} < \frac{1}{k(A)}}$$

$$\frac{\|\delta x\|}{\|x\|} \le \left(\frac{k(A)}{1 - k(A)\frac{\|\delta A\|}{\|A\|}} \right) \frac{\|\delta A\|}{\|A\|}$$

Note Title 3/2/2011

Condition number $= \|A\| \, \|A^{-1}\| = k(A)$

$Ax = b$,

$(A + \delta A)(x + \delta x) = b + \delta b$

$$\boxed{\frac{\|\delta A\|}{\|A\|} < \frac{1}{k(A)}}$$

$$\frac{\|\delta x\|}{\|x\|} \le \frac{k(A)\left(\frac{\|\delta A\|}{\|A\|} + \frac{\|\delta b\|}{\|b\|}\right)}{1 - k(A)\frac{\|\delta A\|}{\|A\|}}$$

$k(A)$ small : well-conditioned,

$k(A)$ large : ill-conditioned.

How large, how small ?

No precise boundary.

depends upon the accuracy of the data, the computer, the amount of error that can be tolerated.

Suppose that $\frac{\|\delta b\|}{\|b\|} \simeq 10^{-4}$.

If $k(A) \leq 10^{2}$, then $\frac{\|\delta x\|}{\|x\|} \leq 10^{-2}$

If $k(A) \simeq 10^{4}$, then in the worst case

$$\frac{\|\delta x\|}{\|x\|} \simeq 1 \quad : \text{not acceptable.}$$

$$k(A) = \|A\| \, \|A^{-1}\|$$

$$\alpha \neq 0, \quad k(\alpha A) = \|\alpha A\| \, \|(\alpha A)^{-1}\|$$

$$= |\alpha| \, \|A\| \, \frac{1}{|\alpha|} \, \|A^{-1}\| = k(A)$$

Let $\epsilon > 0$ and define $A_\epsilon = \begin{bmatrix} \epsilon & 0 \\ 0 & \epsilon \end{bmatrix}$.

Then for any induced matrix norm

$$\|A_\epsilon\| = \epsilon, \quad \|A_\epsilon^{-1}\| = \frac{1}{\epsilon} \quad \text{and} \quad k(A) = 1.$$

$\det(A_\epsilon) = \epsilon^2$: can be made as small as we wish.

No connection between small determinant and ill conditioning.

Ill Conditioning because of poor scaling.

Consider $\begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \epsilon \end{bmatrix}$. $A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\epsilon} \end{bmatrix}$

If $\epsilon << 1$, then $k_1(A) = k_\infty(A) = \frac{1}{\epsilon} >> 1$

Multiply the second equation by $\frac{1}{\epsilon}$:

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$: well-conditioned.

Scaling

Note Title 12/9/2010

$Ax = b$ Multiply an equation by $c \neq 0$

$\Rightarrow$ Same Solution. Row Scaling

Multiply jth column by $c \neq 0$

$\Rightarrow$ Solution is altered only in the jth component.

jth component is multiplied by $\frac{1}{c}$: Exercise.

Column Scaling.

Define $\quad \text{maxmag}(A) = \max_{x \neq 0} \dfrac{\|Ax\|}{\|x\|}$: maximum magnification by A

$\text{minmag}(A) = \min_{x \neq 0} \dfrac{\|Ax\|}{\|x\|}$: minimum magnification by A

Exercise: $\quad k(A) = \dfrac{\text{maxmag}(A)}{\text{minmag}(A)}$.

Maximum magnification much larger than minimum magnification $\Rightarrow$ ill conditioned.

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}, \quad \operatorname{minmag}(A) = \min_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

$$\kappa(A) = \frac{\|A\|}{\operatorname{min\,mag}(A)}. \qquad = \frac{1}{\|A^{-1}\|}.$$

Let vector norm be $\|\cdot\|_1$, $\|\cdot\|_2$ or $\|\cdot\|_\infty$.

Then $\|A\| \geq \|A e_j\| = \|C_j\|$ and

$\operatorname{minmag}(A) \leq \|A e_i\| = \|C_i\|$. Thus

$\kappa(A) \geq \dfrac{\|C_j\|}{\|C_i\|}$: If A has columns which differ by several orders of magnitude, then A : ill-conditioned.

Consider $A = \begin{bmatrix} 1000 & 999 \\ 999 & 998 \end{bmatrix}$. Then $A^{-1} = \begin{bmatrix} -998 & 999 \\ 999 & -1000 \end{bmatrix}$

$\|A\|_\infty = 1999 = \|A^{-1}\|_\infty$ $\quad$ $k(A) \simeq 3.99 \times 10^6$

$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1999 \\ 1997 \end{bmatrix}$. $\quad \dfrac{\|Ax\|_\infty}{\|x\|_\infty} = 1999 = \|A\|_\infty$.

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$: direction of maximum magnification by A

$A^{-1} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1997 \\ 1999 \end{bmatrix}$ $\quad \dfrac{\|A^{-1}x\|_\infty}{\|x\|_\infty} = 1999 = \|A^{-1}\|_\infty$

$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$: direction of maximum magnification by A^{-1}.

$$A^{-1}\begin{bmatrix}-1\\1\end{bmatrix}=\begin{bmatrix}1997\\-1999\end{bmatrix} \Rightarrow A\begin{bmatrix}1997\\-1999\end{bmatrix}=\begin{bmatrix}-1\\1\end{bmatrix}$$

direction of minimum magnification by A.

Ill Conditioning: Geometric Picture

Let A be a ill-conditioned matrix whose rows and columns are not severely out of scale.

Normalize A so that $\|A\| = 1$.

$$1 << k(A) = \frac{1}{\text{minmag}(A)} \Rightarrow \text{minmag}(A) << 1.$$

$\Rightarrow \exists\, x \in \mathbb{R}^n$, $\|x\| = 1$ such that $\|Ax\| << 1$.

$$Ax = \sum_{j=1}^{n} x_j A e_j = \sum_{j=1}^{n} x_j C_j \simeq 0$$

$\Rightarrow$ Colums of A are 'nearly' linearly dependent.

Consider $$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

The solution set of each of the equation is a straight line in the (x_1, x_2) plane.

The first line is perpendicular to $[a_{11}\ a_{12}]$, the second line is perpendicular to $[a_{21}\ a_{22}]$.

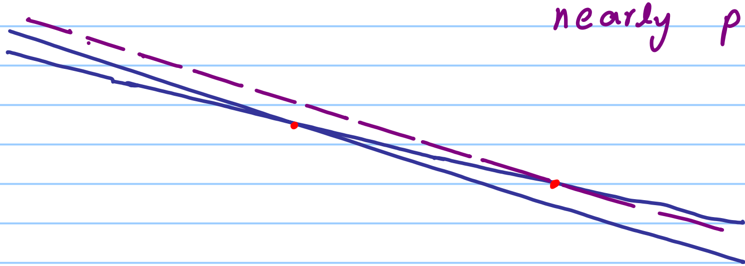
The solution of the system is the point of intersection of these two lines.

Consider $$1000\,x_1 + 999\,x_2 = b_1$$
$$999\,x_1 + 998\,x_2 = b_2 .$$

The slopes of the two lines are

$$m_1 = -\frac{1000}{999} \simeq -1.001001, \quad m_2 = \frac{-999}{998} \simeq -1.001002$$

nearly parallel.



Note Title 11/24/2010

Small Pivots

Consider

well-conditioned

$$\begin{bmatrix} .002 & 1.231 & 2.471 \\ 1.196 & 3.165 & 2.543 \\ 1.475 & 4.271 & 2.142 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3.704 \\ 6.904 \\ 7.888 \end{bmatrix}$$

exact solution: $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

small pivot: .002

$$m_{21} = \frac{1.196}{0.002} = 598.0, \quad m_{31} = \frac{1.475}{0.002} = 737.5$$

$R_2 - m_{21} R_1, \quad R_3 - m_{31} R_1$

$3.165 - (598.0)(1.231)$