Note Title 3/29/2011

Tutorial 3

Q.1 Let $L = [l_{ij}]$, $M = [m_{ij}]$ be unit lower triangular matrices. Show that LM is also a unit lower triangular matrix.

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L : unit lower triangular

$$L = \begin{bmatrix} 1 & & & & \\ l_{21} & 1 & & 0 & \\ l_{31} & l_{32} & 1 & & \\ \vdots & & & \ddots & \\ l_{n1} & l_{n2} & \cdots & l_{n,n-1}, & 1 \end{bmatrix}$$

$l_{ij} = 0$ if $i < j$, $l_{ii} = 1$, $i = 1, \cdots, n$.

$L = [l_{ij}]$, $M = [m_{ij}]$ unit lower triangular

$l_{ij} = 0$, $m_{ij} = 0$ if $i < j$

$l_{ii} = 1$, $m_{ii} = 1$, $i = 1, \cdots, n$

$$LM(i,j) = \sum_{k=1}^{n} l_{ik} m_{kj} = \sum_{k=1}^{i} l_{ik} m_{kj}$$

If $i < j$, then $m_{kj} = 0$ for $k = 1, \cdots, i$

Hence $LM(i,j) = 0$

$$LM(i,i) = \sum_{k=1}^{n} l_{ik} m_{ki} = \sum_{k=1}^{i} l_{ik} m_{ki}$$

$$= l_{ii} m_{ii} = 1 .$$

Computation of A^{-1}.

$$A^{-1} = [A^{-1}e_1, A^{-1}e_2, \cdots, A^{-1}e_j, \cdots, A^{-1}e_n]$$

The jth column of A^{-1} is $C_j = A^{-1}e_j$,

that is, $AC_j = e_j$, $j = 1, \cdots, n$

Q.2. If $L = [l_{ij}]$ is unit lower triangular, then L^{-1} is also unit lower triangular.

Solution: Note that $L^{-1} = [L^{-1}e_1, \cdots, L^{-1}e_j, \cdots, L^{-1}e_n]$

Thus the jth column C_j of L^{-1} is given by

$$C_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix} = L^{-1}e_j$$

Hence $L\,C_j = e_j$

$$L\,c_j = e_j$$

$$\begin{bmatrix} 1 & 0 & & & \\ l_{21} & 1 & 0 & & \\ l_{31} & l_{32} & 1 & & \\ \vdots & & & & \\ l_{n1} & l_{n2} & l_{n3} & \cdots & 1 \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ b_{3j} \\ \vdots \\ b_{nj} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Forward Substitution : $b_{1j} = 0$, $b_{2j} = 0, \cdots,$

$b_{j-1,j} = 0$, jth eq^n: $b_{jj} = 1$

$(j+1)$st eq^n: $l_{j+1,j}\, b_{jj} + b_{j+1,j} = 0 \Rightarrow$

$$b_{j+1,j} = -l_{j+1,j}$$

$$c_j = L^{-1} e_j = \begin{array}{r} \\ \\ \\ j \rightarrow \\ \\ \\ \\ \\ \end{array} \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ 1 \\ -l_{j+1,j} \\ * \\ \vdots \\ * \end{array}\right]$$

$$L^{-1} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ * & 1 & & 0 \\ \vdots & * & & 0 \\ \vdots & \vdots & & \vdots \\ & & & 0 \\ * & * & & 1 \end{bmatrix}$$

unit lower triangular.

U, V : unit upper triangular

$\Rightarrow U^t, V^t$: unit lower triangular

$\Rightarrow V^t U^t = (UV)^t$: unit lower triangular

$\Rightarrow UV$: unit upper triangular

U : unit upper triangular

$\Rightarrow U^t$: unit lower triangular

$\Rightarrow (U^t)^{-1} = (U^{-1})^t$: unit lower triangular

$\Rightarrow U^{-1}$: unit upper triangular

Uniqueness of LU decomposition

$A = L_1 U_1 = L_2 U_2$, where

L_1, L_2 : unit lower triangular,

U_1, U_2 : upper triangular (invertible)

$$L_2^{-1} L_1 = U_2 U_1^{-1} = I$$

← upper triangular

↗ unit lower triangular

Partitioning of a matrix

A : $n \times n$ matrix

Let $1 \leq k \leq n$

$$A = \begin{array}{cc} \left[\begin{array}{cc} A_{11} & A_{12} \\ & \\ A_{21} & A_{22} \end{array}\right] & \begin{array}{l} k \\ \\ n-k \end{array} \\ \begin{array}{cc} k & n-k \end{array} & \end{array}$$

A_{11} : $k \times k$

A_{12} : $k \times (n-k)$

A_{21} : $(n-k) \times k$

A_{22} : $(n-k) \times (n-k)$

$$C = A\,B$$

$$\begin{array}{c} \begin{array}{cc} k & n-k \end{array} \\ \left[\begin{array}{cc} C_{11} & C_{12} \\ C_{21} & C_{22} \end{array}\right] \begin{array}{c} k \\ n-k \end{array} \end{array} = \begin{array}{c} \begin{array}{cc} k & n-k \end{array} \\ \left[\begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array}\right] \end{array} \begin{array}{c} \begin{array}{cc} k & n-k \end{array} \\ \left[\begin{array}{cc} B_{11} & B_{12} \\ B_{21} & B_{22} \end{array}\right] \begin{array}{c} k \\ n-k \end{array} \end{array}$$

Claim: $C_{11} = A_{11} B_{11} + A_{12} B_{21}$

Claim: $C_{11} = A_{11} B_{11} + A_{12} B_{21}$

$A = [a_{ij}]$, $B = [b_{ij}]$, $C = [c_{ij}]$, $C = AB$

$$\begin{array}{cc} \begin{array}{c} k \\ n-k \end{array} & \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \\ & \begin{array}{cc} k & n-k \end{array} \end{array} = \begin{array}{c} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \\ \begin{array}{cc} k & n-k \end{array} \end{array} \begin{array}{cc} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} & \begin{array}{c} k \\ n-k \end{array} \\ \begin{array}{cc} k & n-k \end{array} & \end{array}$$

$$c_{ij} = \sum_{p=1}^{n} a_{ip} b_{pj} = \sum_{p=1}^{k} a_{ip} b_{pj} + \sum_{p=k+1}^{n} a_{ip} b_{pj}$$

$$C_{11}(i,j) = \sum_{p=1}^{k} A_{11}(i,p) B_{11}(p,j) + \sum_{p=k+1}^{n} A_{12}(i,p) B_{21}(p,j)$$

$\det(A_k) \neq 0 \implies A = L\,U$

L : unit lower triangular

U : upper triangular

The converse is true.

Q.3 Let A be an $n \times n$ invertible matrix which can be written as LU. For $k = 1, 2, \cdots, n$, let A_k denote the principal leading submatrix of order k. Show that

$$\det(A_k) \neq 0, \quad k = 1, 2, \cdots, n$$

Solution : $A = LU \implies \det(U) = \det(A) \neq 0$

$\det(U) = u_{11}\, u_{22} \cdots u_{nn} \;:\; u_{ii} \neq 0$

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} \begin{array}{cc} \left[\begin{array}{cc} U_{11} & U_{12} \\ 0 & U_{22} \end{array}\right] & \begin{array}{l} k \\ n-k \end{array} \\ \begin{array}{cc} k & n-k \end{array} & \end{array}$$

$A_{11} = L_{11}\, U_{11}$, i.e., $A_k = L_k U_k$

$$\det(A_k) = \det(U_k) = u_{11} \cdots u_{kk} \neq 0$$

Q.4 Find the LU decomposition of the following matrix.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 8 \\ 3 & 8 & 14 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 8 \\ 3 & 8 & 14 \end{bmatrix} \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{bmatrix}$$

$$\xrightarrow{R_3 - 2R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = U \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

<u>Definition</u> : A is positive-definite if

$A^t = A$ and for $x \neq \bar{0}$, $x^t A x > 0$.

Q.5 If A is positive-definite, then

$a_{ii} > 0$, $i = 1, \cdots, n$

Solution:

$$e_i^t A e_i = [0, \cdots, 0, 1, 0 \cdots 0] \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ \vdots \\ a_{ni} \end{bmatrix} = a_{ii} > 0 .$$

Q. 6 $D = \text{diag.}(d_1, d_2, \cdots, d_n)$ is positive-definite if and only if $d_i > 0$, $i = 1, \cdots, n$

Solution: $\Rightarrow$ $e_i^t D e_i = d_i > 0$, $i = 1, \cdots, n$.

$\Leftarrow$ $D^t = D$.

$$x^t D x = \sum_{j=1}^{n} d_j x_j^2 > 0 \quad \text{if } x \neq \bar{0}$$

Hence D is positive-definite

Consider the first step of Gauss-elimination:

$$m_{i1} = \frac{a_{i1}}{a_{11}}, \quad i = 2, \cdots, n, \quad R_i \rightarrow R_i - m_{i1} R_1$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \longrightarrow \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22}^{(1)} & \cdots & a_{2n}^{(1)} \\ \vdots & & & \\ 0 & a_{n2}^{(1)} & \cdots & a_{nn}^{(1)} \end{bmatrix}$$

$$a_{ij}^{(1)} = a_{ij} - m_{i1} a_{1j}, \quad i, j = 2, \cdots, n$$

$$A \longrightarrow \begin{bmatrix} a_{11} & \tilde{R}_1 \\ 0 & A^{(1)} \end{bmatrix}$$

$$\tilde{R}_1 = [a_{12} \quad a_{13} \cdots \quad a_{1n}]$$

$$A^{(1)} = [a_{ij}^{(1)}] \text{ , } i, j = 2, \cdots, n \ : (n-1) \times (n-1) \text{ matrix}$$

Q.7 If A is symmetric, then $A^{(1)}$ is also symmetric

Solution :

$$a_{ij}^{(1)} = a_{ij} - m_{i1} a_{1j}$$

$$= a_{ij} - \frac{a_{i1}}{a_{11}} a_{1j}$$

$$= a_{ji} - \frac{a_{j1}}{a_{11}} a_{1i}$$

$$= a_{ji} - m_{j1} a_{1i} = a_{ji}^{(1)}$$

Q. 8 Show that if a non-singular system $Ax = b$ is altered by multiplication of its jth column by $\alpha \neq 0$, then the solution is altered only in the jth component, which is multiplied by $\frac{1}{\alpha}$.

Solution: $Ax = b$, $x = [x_1, \cdots, x_n]^t = x_1 e_1 + \cdots + x_n e_n$.

$x_1 Ae_1 + \cdots + x_j Ae_j + \cdots + x_n Ae_n = b$.

Ae_j : jth column

$\tilde{A} = [Ae_1, \cdots, Ae_{j-1}, \alpha Ae_j, Ae_{j+1}, \cdots, Ae_n]$

$x_1 Ae_1 + \cdots + (\frac{1}{\alpha}) x_j (\alpha Ae_j) + \cdots + x_n Ae_n = b$

$\tilde{A}\tilde{x} = b$, $\tilde{x} = [x_1, \cdots, x_{j-1}, \frac{1}{\alpha} x_j, \cdots, x_n]$