Note Title 4/5/2011

Non-linear Equations

$f : [a, b] \longrightarrow \mathbb{R}$

Aim: To find c such that $f(c) = 0$

If $f : [a, b] \longrightarrow \mathbb{R}$ is continuous and $f(a)f(b) \leq 0$, then by the Intermediate value theorem

$$f(c) = 0 \quad \text{for some } c \in [a, b].$$

Bisection Method

Given: $f : [a, b] \longrightarrow \mathbb{R}$ is continuous, $f(a)\,f(b) < 0$ and f has a unique zero, c, in $[a, b]$

Method: Let $m = \frac{a+b}{2}$.

If $f(m) = 0$, then $c = m$. Otherwise either $f(a)\,f(m) < 0$ or $f(m)\,f(b) < 0$.

Choose $a_1 = a$, $b_1 = m$. Choose $a_1 = m$, $b_1 = b$

Bisection Method : Algorithm.

Let $a_0 = a$, $b_0 = b$.

For $n = 0, 1, 2, \cdots$

$$m = \frac{a_n + b_n}{2}$$

If $f(a_n) f(m) < 0$, then set

$$a_{n+1} = a_n, \quad b_{n+1} = m$$

else set $a_{n+1} = m$, $b_{n+1} = b_n$

Thus $a_n \to c$, $b_n \to c$ as $n \to \infty$.

Drawback: The convergence is very slow.

At each step: One more correct binary digit

Example: $f(x) = x^3 - x - 1 \quad , \quad x \in [1, 2]$

f is continuous, $f(1) = -1$, $f(2) = 5$

$f'(x) = 3x^2 - 1 > 0$ for $x \in [1, 2]$

$\Rightarrow$ f is strictly increasing

$\Rightarrow$ f has a unique zero in $[1, 2]$

$$f\left(\frac{3}{2}\right) = \frac{27}{8} - \frac{3}{2} - 1 = \frac{27 - 12 - 8}{8} = \frac{7}{8}$$

$$= 0.875.$$

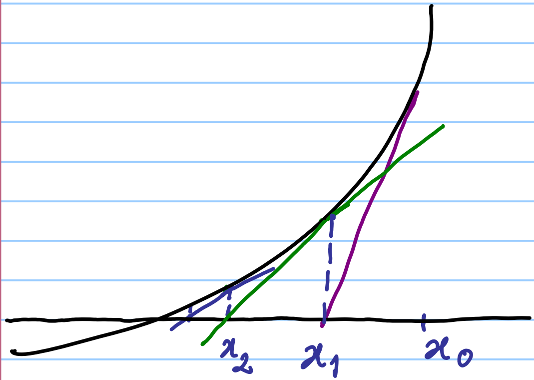
$\Rightarrow$ f has a zero in $[1, 1.5]$ ...

Definition

f has a simple zero at c if

$f(c) = 0$, but $f'(c) \neq 0$.

Newton's method. (simple zero)



Equation of the tangent at x_0

$$\frac{y - f(x_0)}{x - x_0} = f'(x_0)$$

$$y = f(x_0) + f'(x_0)(x - x_0)$$
$$= 0$$

$$\Longrightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

x_0 : given

Newton's Method

x_0 : initial guess.

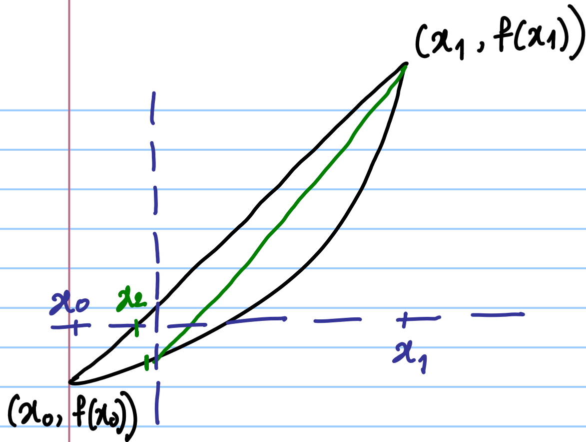
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \cdots$$

Secant Method

x_0, x_1 : given

$$f'(x_n) \simeq \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} = f[x_{n-1}, x_n]$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f[x_{n-1}, x_n]}$$



Equation of the secant through

$(x_0, f(x_0))$ and $(x_1, f(x_1))$

$$\frac{y - f(x_0)}{x - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$y = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

$$y = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) = 0$$

$$\Rightarrow x_1 = x_0 - f(x_0)\frac{x_1 - x_0}{f(x_1) - f(x_0)}$$

$$= x_0 - \frac{f(x_0)}{f[x_0, x_1]}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f[x_{n-1}, x_n]}, \quad n = 1, 2, \cdots$$

Fixed Point

Let $g : [a, b] \longrightarrow [a, b]$. A point $c \in [a,b]$ is said to be a <u>fixed</u> <u>point</u> if

$$g(c) = c.$$

Define

$$f(x) = g(x) - x$$

$$f(c) = 0 \Longleftrightarrow g(c) = c$$

Proposition: Let $g : [a,b] \to [a,b]$ be continuous. Then g has a fixed point c in $[a,b]$.

Proof: If $g(a) = a$, then $c = a$.

If $g(b) = b$, then $c = b$.

Consider the case when $g(a) \neq a$, $g(b) \neq b$.

Then $g(a) > a$, $g(b) < b$.

Define $f(x) = g(x) - x$.

$g(a) > a, \quad g(b) < b \implies$

$f(a) > 0, \quad f(b) < 0$.

$f : [a, b] \longrightarrow \mathbb{R}$, continuous.

Hence by the Intermediate Value Theorem,

$f(c) = 0$ for some $c \in (a, b)$.

$\implies g(c) = c$.

The result is <u>not</u> <u>true</u> if

1) $[a, b]$ is replaced by (a, b).

$g(x) = x^2, \quad x \in (0, 1)$

2) $[a, b]$ is replaced by $[a, \infty)$

$g(x) = x + 1, \quad x \in [0, \infty)$

3) g is not continuous

$$g(x) = \begin{cases} x^2, & x \in (0,1) \\ 1, & x = 0 \\ 0, & x = 1 \end{cases}$$

Consider $g(x) = \frac{x+1}{2}, \quad x \in [0,1]$

Unique fixed point : $c = 1$

$g(x) = x^2, \quad x \in [0,1]$

Two fixed points : $c = 0$ and $c = 1$

$g(x) = x, \quad x \in [0,1]$

Infinitely many fixed points :

Each $c \in [0,1]$ is a fixed point.

Uniqueness of the fixed point

Theorem: If $g : [a,b] \longrightarrow [a,b]$ is continuous and g is differentiable on (a,b) with $|g'(x)| < 1$, $x \in (a,b)$, then g has a unique fixed point in $[a,b]$.

Proof: Existence is already proved.

Let $g(c_1) = c_1$, $g(c_2) = c_2$, $c_1, c_2 \in [a,b]$

Then

$$c_1 - c_2 = g(c_1) - g(c_2)$$

$$= (c_1 - c_2)\, g'(d), \quad d \in (c_1, c_2)$$

by the Mean Value Theorem.

Thus

$$|c_1 - c_2| = |c_1 - c_2|\,|g'(d)|.$$

Since $|g'(d)| < 1$, it follows that $c_1 = c_2$

Picard Fixed Point Iteration

Assumptions: $g : [a, b] \to [a, b]$ is continuous,
g is differentiable on (a, b),
$|g'(x)| \leq K < 1$, $x \in (a, b)$.

Then g has a unique fixed point c in $[a, b]$. Let $x_0 \in [a, b]$. Define

$$x_n = g(x_{n-1}), \quad n = 1, 2, \cdots$$

Then $x_n \to c$ as $n \to \infty$.

Proof: We have

$$|g'(x)| \leq K < 1, \quad x \in (a,b),$$

$$g(c) = c, \quad x_0 \in [a,b], \quad x_n = g(x_{n-1}), \quad n = 1, 2, \cdots$$

Consider

$$x_n - c = g(x_{n-1}) - g(c) = (x_{n-1} - c)\, g'(d_n) \quad \text{MVT}$$

Hence $|x_n - c| \leq K\,|x_{n-1} - c| \leq K^2\,|x_{n-2} - c|$

$$\cdots \leq K^n\,|x_0 - c|$$

$K^n \longrightarrow 0$ as $n \longrightarrow \infty \Rightarrow x_n \longrightarrow c$ as $n \longrightarrow \infty$.

Definition: Let $x_0, x_1, \cdots, x_n, \cdots$ be a sequence which converges to c, and set $e_n = x_n - c$. If there exists a constant $M \neq 0$ and a real number p such that

$$\lim_{n \to \infty} \frac{|e_{n+1}|}{|e_n|^p} = M,$$

then M is called asymptotic error constant and p is called the order of convergence.

Examples : 1) $x_n = \frac{1}{n} \to 0$ as $n \to \infty$.

$$\frac{e_{n+1}}{e_n} = \frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1} \to 1. \qquad p = 1, \; M = 1.$$

2) $x_n = \frac{1}{\sqrt{n}}$, $\quad \frac{e_{n+1}}{e_n} = \frac{\sqrt{n}}{\sqrt{n+1}} \to 1. \qquad p = 1, \; M = 1.$

3) Let $x_0 = \frac{1}{3}$, $x_1 = x_0^2$, $x_2 = x_1^2$, $\dots$, $x_{n+1} = x_n^2$

Then $x_n \to 0$. $\quad \frac{e_{n+1}}{e_n^2} = \frac{x_n^2}{x_n^2} = 1. \qquad p = 2, \; M = 1.$