Note Title 4/6/2011

Order of Convergence

$x_n \longrightarrow c$ as $n \to \infty$.

$e_n = x_n - c$, $\lim\limits_{n\to\infty} \dfrac{|e_{n+1}|}{|e_n|^p} = M \neq 0$

p : order of convergence

M : asymptotic error constant

Fixed Point Iteration : Order of Convergence

$g : [a, b] \longrightarrow [a, b]$, $g(c) = c$: unique fixed point

$x_0 \in [a, b]$, $x_{n+1} = g(x_n)$

Suppose that $g'(c) \neq 0$, $g' \in C[a,b]$

$e_{n+1} = x_{n+1} - c = g(x_n) - g(c) = g'(d_n)(x_n - c)$

d_n between x_n & c

$$\frac{|e_{n+1}|}{|e_n|} = |g'(d_n)| \longrightarrow |g'(c)| = M$$

$$\boxed{p = 1}$$

Newton's Method

f has a simple zero at c :

$$f(c) = 0 , \quad f'(c) \neq 0$$

x_0 : initial guess

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} , \quad n = 0, 1, \cdots .$$

Let $f : [a, b] \longrightarrow \mathbb{R}$.

Assume that f and f' are continuous on $[a, b]$.

Let $x_0 \in [a, b]$.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \in [a, b] \quad \text{for } n = 0, 1, \cdots.$$

If $x_n \longrightarrow d$, then $d = d - \dfrac{f(d)}{f'(d)}$.

$$\Longrightarrow f(d) = 0.$$

$$f : [-3, 3] \to \mathbb{R}$$

$$f(x) = \begin{cases} \sqrt{x-1}, & x \geq 1 \\ -\sqrt{1-x}, & x < 1 . \end{cases} \qquad f(1) = 0$$

$$f'(x) = \begin{cases} \dfrac{1}{2\sqrt{x-1}}, & x > 1 \\ \dfrac{1}{2\sqrt{1-x}}, & x < 1 . \end{cases}$$

$$x_{n+1} = x_n - 2(x_n - 1) = 2 - x_n$$

$x_{n+1} - 1 = 1 - x_n$. : oscillates between x_0 & $1 - x_0$.

Newton's Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \cdots$$

$$g(x) = x - \frac{f(x)}{f'(x)}, \quad x_{n+1} = g(x_n).$$

$f(c) = 0$, $f'(c) \neq 0$. $f(c) = 0 \iff g(c) = c$.

$$g'(x) = 1 - \frac{f'(x)^2 - f(x) f''(x)}{f'(x)^2} = \frac{f(x) f''(x)}{f'(x)^2}$$

Sufficient Conditions : Convergence .

$f \in C^2[a,b]$, $f'(x) \neq 0$, $x \in [a,b]$

$$g(x) = x - \frac{f(x)}{f'(x)} : [a,b] \longrightarrow [a,b]$$

$$|g'(x)| = \left| \frac{f(x) f''(x)}{f'(x)^2} \right| < 1 , \quad x \in (a,b)$$

Let $x_0 \in [a,b]$ and $x_{n+1} = g(x_n)$.

Then $x_n \longrightarrow c$, c : unique zero of f in $[a,b]$

Theorem: Let $f \in C^2[a,b]$.

1) $f(a)f(b) < 0$, 2) $f'(x) \neq 0$, $x \in [a,b]$.

3) $f''(x) \geq 0$ or $f''(x) \leq 0$ on $[a,b]$.

4) $\dfrac{|f(a)|}{|f'(a)|} < b-a$, $\dfrac{|f(b)|}{|f'(b)|} < b-a$

Then for any $x_0 \in [a,b]$, $x_n \to c$ with $f(c) = 0$, $f'(c) \neq 0$.

$$x_0 = a$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = a - \frac{f(a)}{f'(a)}$$

$$|x_1 - a| = \left| \frac{f(a)}{f'(a)} \right| < b - a$$

$$\Rightarrow x_1 \in [a, b]$$

Order of Convergence : Newton's Method .

$f(c) = 0 \ , \quad f'(c) \neq 0$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f(c) = f(x_n) + f'(x_n)(c - x_n) + \frac{f''(d_n)}{2}(c - x_n)^2 .$$

$$x_n - \frac{f(x_n)}{f'(x_n)} - c = \frac{f''(d_n)}{2 f'(x_n)}(c - x_n)^2 .$$

$$\frac{|e_{n+1}|}{|e_n|^2} = \frac{f''(d_n)}{2 f'(x_n)} \longrightarrow \frac{f''(c)}{2 f'(c)}$$

$$\boxed{p = 2}$$

Note Title 3/8/2011

Secant Method

$x_0, x_1 \in [a, b]$ given

$$x_{n+1} = x_n - \frac{f(x_n)}{f[x_{n-1}, x_n]} = x_n - \frac{f(x_n)}{\frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}}$$

$$x_n \longrightarrow c \implies f[x_{n-1}, x_n] \longrightarrow f[c, c] = f'(c).$$

Thus $$c = c - \frac{f(c)}{f'(c)} \implies f(c) = 0.$$

Secant Method: $x_{n+1} = x_n - \dfrac{f(x_n)}{f[x_n, x_{n-1}]}$

$$= x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$$

$$= \frac{x_n \cancel{f(x_n)} - x_n f(x_{n-1}) - \cancel{x_n f(x_n)} + x_{n-1} f(x_n)}{f(x_n) - f(x_{n-1})}$$

$$= \frac{x_{n-1}(f(x_n) - f(x_{n-1})) - f(x_{n-1})(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$$

$$= x_{n-1} - \frac{f(x_{n-1})}{f[x_n, x_{n-1}]}$$

Error in the Secant Method.

$$f(x) = f(x_n) + f[x_n, x_{n-1}](x - x_n) + f[x_n, x_{n-1}, x](x - x_n)(x - x_{n-1})$$

$$0 = f(c) = f(x_n) + (c - x_n)\frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} + f[x_n, x_{n-1}, c](c - x_n)(c - x_{n-1})$$

$$x_n - f(x_n)\frac{(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} - c = \frac{f[x_n, x_{n-1}, c]}{f[x_n, x_{n-1}]}(c - x_n)(c - x_{n-1})$$

$$|c - x_{n+1}| = \left|\frac{f''(\alpha_n)}{2f'(r_n)}\right| |c - x_n|\,|c - x_{n-1}|$$