Note Title 4/7/2011

Error in the Secant Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f[x_n, x_{n-1}]}, \quad n = 1, 2, \cdots$$

$$0 = f(c) = f(x_n) + f[x_n, x_{n-1}](c - x_n)$$

$$+ f[x_n, x_{n-1}, c](c - x_n)(c - x_{n-1})$$

$$\Rightarrow x_n - \frac{f(x_n)}{f[x_n, x_{n-1}]} - c = \frac{f[x_n, x_{n-1}, c]}{f[x_n, x_{n-1}]} e_n \, e_{n-1}$$

$$\Rightarrow x_{n+1} - c = \frac{f''(\alpha_n)}{2 f'(\gamma_n)} e_n \, e_{n-1} \qquad e_n = c - x_n$$

$$|e_{n+1}| = \frac{|f''(d_n)|}{2|f'(r_n)|}|e_n||e_{n-1}| = \alpha_n |e_n||e_{n-1}|$$

$$\lim_{n\to\infty} \alpha_n = \frac{|f''(c)|}{2|f'(c)|}$$

It can be shown that the order of convergence $p = 1.618$:

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^p} = M \neq 0$$

Secant Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f[x_n, x_{n-1}]} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$$

$$= \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

$f(x_n)$, $f(x_{n-1})$ can be of the same sign.

Prone to round-off errors.

Regula-falsi Method.

$f : [a, b] \longrightarrow \mathbb{R}$ Continuous, $f(a) f(b) < 0$.

Set $a_0 = a$, $b_0 = b$

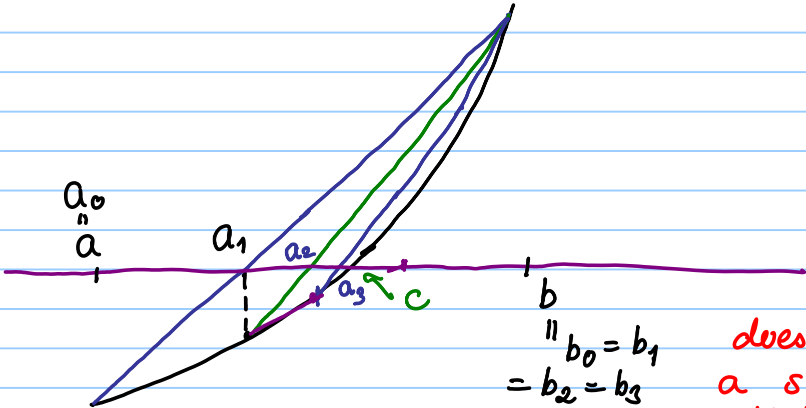
for $n = 0, 1, 2, \cdots$

$$w = \frac{a_n f(b_n) - b_n f(a_n)}{f(b_n) - f(a_n)} \quad : \text{Intersection of the Secant with the } x\text{-axis}$$

If $f(a_n) f(w) < 0$, then $a_{n+1} = a_n$, $b_{n+1} = w$,

else $a_{n+1} = w$, $b_{n+1} = b_n$

$f'(x) > 0,\ f''(x) \geq 0.$



does not give a small interval which contains zero

$w < c$: always $\Rightarrow b_{n+1} = b$.

Note Title 4/7/2011

Newton's Method : $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

2 function evaluations / step

quadratic convergence , derivatives involved

Secant Method : $x_{n+1} = x_n - \frac{f(x_n)}{f[x_n, x_{n-1}]}$

1 function evaluation / step

$p = 1.618$

Order of Convergence : Newton's Method

$$g(x) = x - \frac{f(x)}{f'(x)}, \quad x_{n+1} = g(x_n)$$

$$f(c) = 0, \quad f'(c) \neq 0$$

$$g'(x) = \frac{f(x) f''(x)}{f'(x)^2} \Rightarrow g'(c) = 0$$

$$x_{n+1} - c = g(x_n) - g(c) = (x_n - c) g'(c) + \frac{(x_n - c)^2}{2} g''(d_n)$$

$$e_{n+1} = \frac{g''(d_n)}{2} e_n^2$$

c : double zero of f.

$f(c) = f'(c) = 0, \quad f''(c) \neq 0$ linear convergence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

$$g(x) = x - \frac{f(x)}{f'(x)}. \quad g'(x) = \frac{f''(x)\, f(x)}{f'(x)^2}.$$

$$\lim_{x \to c} g'(x) = \lim_{x \to c} f''(x) \lim_{x \to c} \frac{f(x)}{f'(x)^2}$$

$$= f''(c) \lim_{x \to c} \frac{f'(x)}{2 f'(x) f''(x)} \quad : \text{L'Hospital's Rule}$$

$$= \frac{1}{2}$$

$f(c) = f'(c) = 0$, $f''(c) \neq 0$. Consider.

$$g(x) = x - \frac{2f(x)}{f'(x)}.$$

$$g'(x) = 1 - \frac{2f'(x)^2 - 2f(x)f''(x)}{f'(x)^2}$$

$$= -1 + \frac{2f(x)f''(x)}{f'(x)^2}$$

$$\lim_{x \to c} g'(x) = -1 + 2 \cdot \frac{1}{2} = 0.$$

$$f(c) = f'(c) = 0 \,, \quad f''(c) \neq 0$$

$$x_{n+1} = x_n - \frac{2\, f(x_n)}{f'(x_n)} \,, \quad n = 0, 1, 2, \ldots$$

Quadratic Convergence

Iterative Methods

$Ax = b$, $A = [a_{ij}]$: $n \times n$ invertible

$a_{ii} \neq 0$, $i = 1, 2, \cdots, n$.

$$\sum_{j=1}^{n} a_{ij} x_j = b_i , \quad i = 1, 2, \cdots, n$$

$$x_i = \frac{b_i - \sum\limits_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} x_j}{a_{ii}} , \quad i = 1, \cdots, n$$

$Ax = b$. Exact solution satisfies

$$x_i = \frac{b_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} x_j}{a_{ii}}, \quad i = 1, \cdots, n.$$

Jacobi Method

$x^{(0)} = [x_1^{(0)}, \cdots, x_n^{(0)}]^t$: Initial approx.

$$x_i^{(k)} = \frac{b_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} x_j^{(k-1)}}{a_{ii}}, \quad i = 1, \cdots, n, \quad k = 1, 2, \cdots$$

$$x_i - x_i^{(k)} = \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{a_{ij}}{a_{ii}} \left(x_j - x_j^{(k-1)} \right),$$

$$k = 1, 2, \dots$$

$$\| x - x^{(k)} \|_\infty = \max_{1 \le i \le n} | x_i - x_i^{(k)} |$$

$$| x_i - x_i^{(k)} | \le \| x - x^{(k-1)} \|_\infty \sum_{\substack{j=1 \\ j \neq i}}^{n} \left| \frac{a_{ij}}{a_{ii}} \right|$$

$$\Rightarrow \| x - x^{(k)} \|_\infty \le \Big(\max_{1 \le i \le n} \underbrace{\sum_{\substack{j=1 \\ j \neq i}}^{n} \left| \frac{a_{ij}}{a_{ii}} \right|}_{= \mu} \Big) \| x - x^{(k-1)} \|_\infty$$

$$\|x - x^{(k)}\|_\infty \le \mu \, \|x - x^{(k-1)}\|_\infty \le \cdots$$
$$\le \mu^k \, \|x - x^{(0)}\|_\infty ,$$

$$\mu = \max_{1 \le i \le n} \sum_{\substack{j=1 \\ j \ne i}}^{n} \left| \frac{a_{ij}}{a_{ii}} \right|$$

exact solution

If $\mu < 1$, then $x^{(k)} \longrightarrow x$ as $k \longrightarrow \infty$.

$$\sum_{\substack{j=1 \\ j \ne i}}^{n} |a_{ij}| < |a_{ii}|$$: <u>strictly diagonally row-dominant</u>

Gauss - Seidel Method

$$x_i = \frac{b_i - \sum\limits_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} x_j}{a_{ii}}, \quad i = 1, \cdots, n$$

$$x_i = \frac{b_i - \sum\limits_{j=1}^{i-1} a_{ij} x_j - \sum\limits_{j=i+1}^{n} a_{ij} x_j}{a_{ii}}, \quad i = 1, \cdots, n$$

$$x_i = \frac{b_i - \sum_{j=1}^{i-1} a_{ij} x_j - \sum_{j=i+1}^{n} a_{ij} x_j}{a_{ii}}$$

$$x_i^{(k)} = \frac{b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k-1)}}{a_{ii}},$$

$i = 1, \cdots, n$.

$$\underbrace{x_i - x_i^{(k)}}_{e_i^{(k)}} = -\sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} (x_j - x_j^{(k)}) - \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} (x_j - x_j^{(k-1)})$$

$$e_i^{(k)} = -\sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} e_j^{(k)} - \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} e_j^{(k-1)}$$

Let $\alpha_i = \sum_{j=1}^{i-1} \left| \frac{a_{ij}}{a_{ii}} \right|$, $\beta_i = \sum_{j=i+1}^{n} \left| \frac{a_{ij}}{a_{ii}} \right|$,

$\alpha_1 = 0$, $\beta_n = 0$

$$|e_i^{(k)}| \le \alpha_i \|e^{(k)}\|_\infty + \beta_i \|e^{(k-1)}\|_\infty, \quad i = 1, \cdots, n$$