Note Title 4/11/2011

Gauss - Seidel Method

$$x_i = \frac{b_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} x_j}{a_{ii}} , \quad i = 1, \cdots, n$$

$$x_i = \frac{b_i - \sum_{j=1}^{i-1} a_{ij} x_j - \sum_{j=i+1}^{n} a_{ij} x_j}{a_{ii}} , \quad i = 1, \cdots, n$$

$$x_i = \frac{b_i - \sum_{j=1}^{i-1} a_{ij} x_j - \sum_{j=i+1}^{n} a_{ij} x_j}{a_{ii}}$$

$$x_i^{(k)} = \frac{b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k-1)}}{a_{ii}}, \quad i = 1, \cdots, n.$$

$$\underbrace{x_i - x_i^{(k)}}_{e_i^{(k)}} = -\sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} (x_j - x_j^{(k)}) - \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} (x_j - x_j^{(k-1)})$$

$$e_i^{(k)} = -\sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} e_j^{(k)} - \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} e_j^{(k-1)}$$

Let $\alpha_i = \sum_{j=1}^{i-1} \left| \frac{a_{ij}}{a_{ii}} \right|$, $\beta_i = \sum_{j=i+1}^{n} \left| \frac{a_{ij}}{a_{ii}} \right|$,

$\alpha_1 = 0$, $\beta_n = 0$

$$|e_i^{(k)}| \le \alpha_i \, \|e^{(k)}\|_\infty + \beta_i \, \|e^{(k-1)}\|_\infty , \quad i = 1, \cdots, n$$

$$|e_i^{(k)}| \leq \alpha_i \, \|e^{(k)}\|_\infty + \beta_i \, \|e^{(k-1)}\|_\infty, \quad i = 1, \cdots, n.$$

$$\|e^{(k)}\|_\infty = |e_m^{(k)}|$$

$$\Rightarrow \|e^{(k)}\|_\infty = |e_m^{(k)}| \leq \alpha_m \|e^{(k)}\|_\infty + \beta_m \|e^{(k-1)}\|_\infty$$

$$\Rightarrow \|e^{(k)}\|_\infty \leq \frac{\beta_m}{1-\alpha_m} \|e^{(k-1)}\|_\infty$$

$$\|e^{(k)}\|_\infty \le \frac{\beta_m}{1-\alpha_m}\|e^{(k-1)}\|_\infty \quad \text{for } \underline{\text{some } m}$$

$$\alpha_i = \sum_{j=1}^{i-1}\left|\frac{a_{ij}}{a_{ii}}\right|, \qquad \beta_i = \sum_{j=i+1}^{n}\left|\frac{a_{ij}}{a_{ii}}\right|,$$

$$\alpha_1 = \beta_n = 0 .$$

$$\eta = \max_{1\le i\le n}\frac{\beta_i}{1-\alpha_i} < 1 .$$

$$\|e^{(k)}\|_\infty \le \eta\,\|e^{(k-1)}\|_\infty \le \cdots \le \eta^k\|e^{(0)}\|_\infty$$

Jacobi Method : $\mu = \max_{1 \le i \le n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \left| \frac{a_{ij}}{a_{ii}} \right|$

Gauss-Seidel Method.

$$\eta = \max_{1 \le i \le n} \frac{\beta_i}{1 - \alpha_i}, \quad \beta_i = \sum_{j=1}^{i-1} \left| \frac{a_{ij}}{a_{ii}} \right|,$$

$$\alpha_i = \sum_{j=i+1}^{n} \left| \frac{a_{ij}}{a_{ii}} \right|.$$

Claim: $\mu < 1 \Rightarrow \eta < 1$.

$\mu = \max_i (\alpha_i + \beta_i)$

$$\mu = \max_{1 \le i \le n} (\alpha_i + \beta_i), \quad \eta = \max_{1 \le i \le n} \frac{\beta_i}{1-\alpha_i},$$

$$\alpha_i = \sum_{j=1}^{i-1} \left| \frac{a_{ij}}{a_{ii}} \right|, \qquad \beta_i = \sum_{j=i+1}^{n} \left| \frac{a_{ij}}{a_{ii}} \right|.$$

Claim: $\mu < 1 \Rightarrow \eta \le \mu < 1$

Consider $\alpha_i + \beta_i - \dfrac{\beta_i}{1-\alpha_i} = \dfrac{\alpha_i - \alpha_i^2 - \beta_i \alpha_i}{1-\alpha_i}$

$$= \frac{\alpha_i(1-(\alpha_i+\beta_i))}{1-\alpha_i} \geqslant \frac{\alpha_i}{1-\alpha_i}(1-\mu) > 0$$

$$\mu = \max_{1 \le i \le n} \sum_{\substack{j=1 \\ j \ne i}}^{n} \frac{|a_{ij}|}{|a_{ii}|} < 1$$

$$\Rightarrow \text{For } i = 1, \cdots, n, \quad \sum_{\substack{j=1 \\ j \ne i}}^{n} |a_{ij}| < |a_{ii}|, \text{ i.e.,}$$

A is strictly row-diagonally dominant

$$\begin{bmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 2 \\ 3 \end{bmatrix}$$

exact solution $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

Jacobi Iterates: $x^{(0)} = \overline{0}$

$$x_i^{(1)} = \frac{b_i}{4}, \quad i = 1, 2, 3, 4$$

$$x_1^{(2)} = \frac{3 + x_2^{(1)}}{4}, \quad x_2^{(2)} = \frac{2 + x_1^{(1)} + x_3^{(1)}}{4}, \quad \ldots$$

$$\begin{bmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 2 \\ 3 \end{bmatrix}$$

exact solution $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

Jacobi Iterates: $x^{(0)} = \overline{0}$

$$x_i^{(1)} = \frac{b_i}{4}, \quad i = 1, 2, 3, 4$$

Gauss-Seidel:

$$x_1^{(1)} = \frac{b_1}{4}, \quad x_2^{(1)} = \frac{b_2 + x_1^{(1)} + x_2^{(0)}}{4}, \quad x_3^{(1)} = \frac{b_3 + x_2^{(1)} + x_4^{(0)}}{4}$$

$$\begin{bmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 2 \\ 3 \end{bmatrix}$$

exact solution $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

Jacobi Iterates: $x^{(0)} = \bar{0}$

$$x_1^{(1)} = \frac{3}{4}, \quad x_2^{(1)} = \frac{2}{4}, \quad x_3^{(1)} = \frac{2}{4}, \quad x_4^{(1)} = \frac{3}{4}$$

$$x_1^{(2)} = \frac{3 + \frac{1}{2}}{4} = \frac{7}{8}, \quad x_2^{(2)} = \frac{2 + \frac{3}{4} + \frac{2}{4}}{4} = \frac{13}{16}$$

$$= x_4^{(2)} \qquad\qquad = x_3^{(2)}$$

$$\begin{bmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 2 \\ 3 \end{bmatrix}$$

exact solution $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

<u>Gauss-Seidel</u> <u>Iterates</u> : $x^{(0)} = \bar{0}$

$$x_1^{(1)} = \frac{3}{4}, \quad x_2^{(1)} = \frac{2 + x_1^{(1)} + x_2^{(0)}}{4} = \frac{11}{16}$$

$$x_3^{(1)} = \frac{2 + x_2^{(1)} + x_4^{(0)}}{4} = \frac{2 + \frac{11}{16}}{4} = \frac{43}{64}$$

$$x_4^{(1)} = \frac{3 + x_3^{(1)}}{4} = \frac{3 + \frac{43}{64}}{4} = \frac{235}{256}$$

Exact $\qquad$ Jacobi : $x^{(1)}$ $\qquad$ Gauss-Seidel $x^{(1)}$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{3}{4} \end{bmatrix} = \begin{bmatrix} 0.75 \\ 0.5 \\ 0.5 \\ 0.75 \end{bmatrix}, \quad \begin{bmatrix} \frac{3}{4} \\ \frac{11}{16} \\ \frac{43}{64} \\ \frac{235}{256} \end{bmatrix} \simeq \begin{bmatrix} 0.75 \\ 0.6875 \\ 0.6719 \\ 0.918 \end{bmatrix}.$$

Jacobi Method.

$$x_i^{(k)} = \frac{b_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} x_j^{(k-1)}}{a_{ii}} \quad , \quad i = 1, \cdots, n.$$

A : upper triangular, $a_{ij} = 0$ if $i > j$.

$$x_n^{(1)} = \frac{b_n}{a_{nn}}$$: exact value

$$x_{n-1}^{(2)} = \frac{b_{n-1} - a_{n-1,n}\, x_n^{(1)}}{a_{n-1,n-1}} \quad : \text{ exact value.}$$

In n iterates we obtain the exact solution.

Q. Let $g : [a, b] \longrightarrow [a, b]$ be continuously differentiable and $M = \max_{x \in [a,b]} |g'(x)| < 1$.

Let c be the unique fixed point of g in $[a, b]$: $g(c) = c$. Let $x_0 \in [a, b]$ and

$$x_{n+1} = g(x_n), \quad n = 0, 1, 2, \cdots$$

Show that

$$|x_{n+1} - c| \leq \frac{M}{1-M} |x_{n+1} - x_n|$$

<u>Solution:</u> $g \in C^1[a,b]$, $g(c) = c$.

$|g'(x)| \leq M < 1$

$$x_{n+1} - c = g(x_n) - g(c)$$
$$= g'(d_n)(x_n - c)$$

$$\Rightarrow |x_{n+1} - c| \leq M\,|x_n - c|$$
$$\leq M\,|x_n - x_{n+1}| + M\,|x_{n+1} - c|$$

$$\Rightarrow |x_{n+1} - c| \leq \frac{M}{1-M}\,|x_n - x_{n+1}|$$

Q.5 Let c be the smallest positive root of $f(x) = 20x^3 - 20x^2 - 25x + 4$.

Consider $g(x) = x^3 - x^2 - \frac{x}{4} + \frac{1}{5}$, $x \in [0,1]$.

Then $f(c) = 0 \iff g(c) = c$.

Let $x_0 = 0$, $x_{n+1} = g(x_n)$, $n = 0, 1, 2, \ldots$

Find n such that

$$|c - x_n| < 10^{-3}$$

Solution : $f(x) = 20x^3 - 20x^2 - 25x + 4$

$$= 20x^3 - 20x^2 - 5x + 4 - 20x$$

$$g(x) = x^3 - x^2 - \frac{x}{4} + \frac{1}{5}$$

$$= \frac{f(x)}{20} + x$$

$f(c) = 0 \iff g(c) = c$.

$f(x) = 20x^3 - 20x^2 - 25x + 4$

$f(-1) = -11$, $f(0) = 4$, $f(1) = -21$, $f(2) = 34$

f has roots in

$(-1, 0)$, $(0, 1)$, $(1, 2)$

$c \in (0, 1)$

$$g(x) = x^3 - x^2 - \frac{x}{4} + \frac{1}{5}, \quad x \in [0,1]$$

$$g'(x) = 3x^2 - 2x - \frac{1}{4}$$

We want to find $M = \max\limits_{x \in [0,1]} |g'(x)|$

$$g''(x) = 6x - 2 = 0 \Rightarrow x = \frac{2}{3}.$$

x	0	$\frac{2}{3}$	1
$g'(x)$	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{3}{4}$

$$M = \frac{3}{4}$$

$$|c - x_n| \le M^n |c - x_0| \le \left(\frac{3}{4}\right)^n < 10^{-3}$$

$$\Rightarrow \quad 10^3 < \left(\frac{4}{3}\right)^n$$