Note Title 2/26/2012

# Polynomial Interpolation :

Most important topic. Many results will be based on this topic

Numerical Integration / differentiation

Solution of IVP / BVP

Root finding

Note Title 2/25/2012

# Interpolation

$f : [a, b] \longrightarrow \mathbb{R}$ continuous

$x_0, x_1, \cdots, x_n$ : distinct points in $[a, b]$

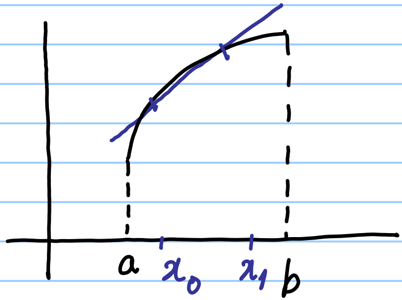
To find a polynomial $p_n$ of degree $\leq n$ such that

$$p_n(x_j) = f(x_j), \quad j = 0, 1, \cdots, n$$

$n = 0$ : $p_n(x) = f(x_0)$ : constant polynomial

$p_n(x_0) = f(x_0)$

$n = 1$,




fitting a straight line

$n = 2$ : fitting a parabola

## General Case:

$f : [a, b] \longrightarrow \mathbb{R}$ continuous

$x_0, x_1, \cdots, x_n$ : distinct point in $[a, b]$

Define

$$l_i(x) = \frac{(x - x_0) \cdots (x - x_{i-1})(x - x_{i+1}) \cdots (x - x_n)}{(x_i - x_0) \cdots (x_i - x_{i-1})(x_i - x_{i+1}) \cdots (x_i - x_n)}$$

Lagrange polynomial : degree $n$

$l_i(x_i) = 1, \quad l_i(x_j) = 0$ for $j \neq i$

# Interpolating Polynomial : Existence

$$l_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{(x - x_j)}{(x_i - x_j)}, \quad l_i(x_j) = \begin{cases} 1, & j = i, \\ 0, & j \neq i. \end{cases}$$

$l_i$ : polynomial of degree $n$

Let

$$p_n(x) = \sum_{i=0}^{n} f(x_i)\, l_i(x).$$

Then $p_n(x_j) = f(x_j), \quad j = 0, 1, \ldots, n$

Let $p_n$ be a polynomial of degree $n$.
Then by the Fundamental theorem of algebra, $p_n(z_1) = 0$ for some $z_1 \in \mathbb{C}$.

$$p_n(x) = (x - z_1)\, q_{n-1}(x)$$

$q_{n-1}$ : polynomial of degree $n-1$

$$p_n(x) = \alpha\, (x - z_1)^{m_1} \cdots (x - z_k)^{m_k},$$

$$m_1 + \cdots + m_k = n.$$

Factorization Thm.

A polynomial of degree $n$ has exactly $n$ zeroes, counted according to their multiplicities.

A non-zero polynomial of degree $\leq n$ has at most $n$ distinct zeroes.

If a polynomial of degree $\leq n$ has <u>more than</u> $n$ zeroes, then it is a zero polynomial.

## Interpolating Polynomial: Uniqueness

$f : [a, b] \longrightarrow \mathbb{R}$ continuous

$x_0, x_1, \cdots, x_n$ : distinct points

Let $p_n$ and $q_n$ be polynomials of degree $\leq n$ such that

$$p_n(x_j) = f(x_j) = q_n(x_j), \; j = 0, 1, \cdots, n$$

$$\Rightarrow (p_n - q_n)(x_j) = 0 \Rightarrow p_n(x) = q_n(x)$$

Theorem : Let $f : [a,b] \to \mathbb{R}$ be continuous.

$x_0, x_1, \cdots, x_n$ : $n+1$ distinct points in $[a,b]$. Then there exists a unique polynomial $p_n$ of degree $\leq n$ such that

$$p_n(x_j) = f(x_j), \quad j = 0, 1, \cdots, n$$

Corollary: $f = q_m$ : polynomial of

of degree $m < n$,

$x_0, x_1, \cdots, x_n$ : distinct points in $[a, b]$

$p_n$ : interpolating polynomial of degree $\leq n$

$$p_n(x_j) = f(x_j), \quad j = 0, 1, \cdots, n$$

$\Longrightarrow p_n = q_m$

Lagrange form of the interpolating polynomial

$$p_n(x) = \sum_{i=0}^{n} f(x_i)\, l_i(x), \quad l_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{(x-x_j)}{(x_i-x_j)}$$

$$p_n(x_j) = f(x_j), \quad j = 0, 1, \cdots, n$$

$p_{n+1}$ : interpolates $f$ at $x_0, \cdots, x_n$ & $x_{n+1}$

$$p_{n+1}(x) = \sum_{i=0}^{n+1} f(x_i)\, \tilde{l}_i(x),$$

$$\tilde{l}_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n+1} \frac{(x-x_j)}{(x_i-x_j)}$$

not recursive

# Divided Difference : Definition

$f : [a,b] \longrightarrow \mathbb{R}$,

$x_0, x_1, \cdots, x_n$ : $n+1$ distinct points in $[a,b]$

$P_n$ : unique interpolating polynomial.

Define the divided difference

$f[x_0\ x_1 \cdots x_n]$ = coefficient of $x^n$ in $P_n(x)$

## Properties of the divided difference

$f[x_0\ x_1 \cdots x_n]$ = coefficient of $x^n$ in $p_n(x)$

1. independent of the order of

   $x_0, x_1, \cdots, x_n$.

2. If $f$ is a polynomial of degree $m < n$, then $p_n(x) = f(x)$ and

   $$f[x_0\ x_1 \cdots x_n] = 0$$

## Recurrence Relation

Let $p_{n-1}$ and $q_{n-1}$ be polynomials of degree $\leq n-1$ such that

$$p_{n-1}(x_j) = f(x_j), \quad j = 0, 1, \cdots, n-1,$$

$$q_{n-1}(x_j) = f(x_j), \quad j = 1, 2, \cdots, n.$$

Consider

$$p_n(x) = \frac{(x - x_0)\, q_{n-1}(x) + (x_n - x)\, p_{n-1}(x)}{x_n - x_0}$$

$$\left.\begin{array}{ll} x_0 & f(x_0) \\ x_1 & f(x_1) \\ \vdots & \vdots \\ x_{n-1} & f(x_{n-1}) \end{array}\right\} p_{n-1}(x)$$

$$\left.\begin{array}{ll} x_1 & f(x_1) \\ \vdots & \vdots \\ x_{n-1} & f(x_{n-1}) \\ x_n & f(x_n) \end{array}\right\} q_{n-1}(x)$$

$$p_n(x_0) = p_{n-1}(x_0) = f(x_0)$$

$$p_n(x_n) = q_{n-1}(x_n) = f(x_n)$$

Define

$$p_n(x) = \frac{(x - x_0)\, q_{n-1}(x) + (x_n - x)\, p_{n-1}(x)}{(x_n - x_0)}$$

$$p_{n-1}(x_j) = f(x_j), \quad j = 0, 1, \cdots, n-1$$

$$q_{n-1}(x_j) = f(x_j), \quad j = 1, 2, \cdots, n$$

$$p_n(x) = \frac{(x - x_0)\, q_{n-1}(x) + (x_n - x)\, p_{n-1}(x)}{x_n - x_0}$$

For $j = 1, \cdots, n-1$,

$$p_n(x_j) = \frac{(x_j - x_0)\, f(x_j) + (x_n - x_j)\, f(x_j)}{x_n - x_0}$$

$$= f(x_j)$$

$$p_n(x) = \frac{(x-x_0)\, q_{n-1}(x) + (x_n - x)\, p_{n-1}(x)}{x_n - x_0}$$

$p_n$ : interpolates $f$ at $x_0, x_1, \dots, x_n$,

$q_{n-1}$ : interpolates $f$ at $x_1, \dots, x_n$,

$p_{n-1}$ : interpolates $f$ at $x_0, \dots, x_{n-1}$

Coeff. of $x^n$ in $p_n(x)$

$$= \frac{\text{Coeff. of } x^{n-1} \text{ in } q_{n-1}(x) - \text{Coeff. of } x^{n-1} \text{ in } p_{n-1}(x)}{x_n - x_0}$$

$$\text{Coeff. of } x^n \text{ in } p_n(x) = \frac{\text{Coeff. of } x^{n-1} \text{ in } q_{n-1}(x) - \text{Coeff. of } x^{n-1} \text{ in } p_{n-1}(x)}{x_n - x_0}$$

$$f[x_0, x_1, \cdots, x_n] = \frac{f[x_1, \cdots x_n] - f[x_0, \cdots, x_{n-1}]}{x_n - x_0}$$

## Formula for $f[x_0, x_1, \cdots, x_n]$

$$p_n(x) = \sum_{i=0}^{n} f(x_i)\, l_i(x), \quad l_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{(x-x_j)}{(x_i-x_j)}$$

$$l_i(x_j) = \begin{cases} 1, & j=i, \\ 0, & j \neq i. \end{cases}$$

Define $\omega(x) = (x-x_0)(x-x_1)\cdots(x-x_n)$

$$\Rightarrow \omega'(x) = (x-x_1)\cdots(x-x_n) + (x-x_0)(x-x_2)\cdots(x-x_n) + \cdots$$

$\Rightarrow \omega'(x_i) = \prod_{\substack{j=0 \\ j\neq i}}^{n} (x_i - x_j)$ and coeff. of $x^n$ in $l_i(x) = \dfrac{1}{\omega'(x_i)}$

$$p_n(x) = \sum_{i=0}^{n} f(x_i)\, l_i(x),$$

Coefficient of $x^n$ in $l_i(x) = \dfrac{1}{\omega'(x_i)}$

where $\omega(x) = \prod_{j=0}^{n} (x - x_j)$

$\Rightarrow$ Coefficient of $x^n$ in $p_n(x) = \sum_{i=0}^{n} \dfrac{f(x_i)}{\omega'(x_i)}$.

4) $$\boxed{f[x_0, x_1, \cdots, x_n] = \sum_{i=0}^{n} \frac{f(x_i)}{\omega'(x_i)}}$$