Note Title 4/5/2011

Tutorial 4

Q. Let $A = [a_{ij}]$ be an $n \times n$ matrix. Define

$$\|A\|_{max} = \max_{1 \le i,j \le n} |a_{ij}| .$$

Does the following property hold?

$$\|AB\|_{max} \le \|A\|_{max} \|B\|_{max}$$

Solution: $\|A\|_{max} = \max_{1\le i,j\le n} |a_{ij}|$.

1) $\|A\|_{max} \geq 0$, $A = 0 \Rightarrow \|A\|_{max} = 0$

$\|A\|_{max} = 0 \Rightarrow a_{ij} = 0$, $1 \le i,j \le n \Rightarrow A = 0$

2) $\alpha A = [\alpha a_{ij}]$, $\|\alpha A\|_{max} = \max_{i,j} |\alpha a_{ij}|$

$= |\alpha| \, \|A\|_{max}$

3) $A + B = [a_{ij} + b_{ij}]$, $\|A+B\|_{max} = \max_{i,j} |a_{ij} + b_{ij}|$

$\leq \max |a_{ij}| + \max |b_{ij}| = \|A\|_{max} + \|B\|_{max}$

$$C = AB$$

$$C_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$\|A\|_{max} = 1, \quad \|A^2\|_{max} = 2$$

$$\|A^2\|_{max} \not\leq \|A\|_{max}^2$$

Q.2. Let A be an invertible matrix.

Define $\operatorname{minmag}(A) = \min_{x \neq \bar{0}} \dfrac{\|Ax\|}{\|x\|}$.

Show that $\operatorname{minmag}(A) = \dfrac{1}{\|A^{-1}\|}$.

<u>Solution:</u> $\operatorname{minmag}(A) = \min_{x \neq \bar{0}} \dfrac{\|Ax\|}{\|x\|}$

$$= \min_{x \neq \bar{0}} \frac{1}{\frac{\|x\|}{\|Ax\|}} = \frac{1}{\max_{x \neq \bar{0}} \frac{\|x\|}{\|Ax\|}}.$$

A invertible: $x \neq \bar{0} \iff y = Ax \neq \bar{0}$

$$\operatorname{min\,mag}(A) = \frac{1}{\max_{y \neq \bar{0}} \frac{\|A^{-1}y\|}{\|y\|}} = \frac{1}{\|A^{-1}\|}$$

$$\text{Condition number} = \|A\|\,\|A^{-1}\|$$

$$= \frac{\displaystyle\max_{x \neq \bar{0}} \frac{\|Ax\|}{\|x\|}}{\dfrac{1}{\|A^{-1}\|}}$$

$$= \frac{\text{max mag}(A)}{\text{min mag}(A)}$$

Q.3 Consider $Ax = b$, where A is invertible.
Let $\hat{x}$ be an approximate solution
and $r = b - A\hat{x}$, the residual.
Show that
$$\frac{\|r\|}{\|A\| \|A^{-1}\|} \leq \frac{\|x - \hat{x}\|}{\|x\|} \leq (\|A\| \|A^{-1}\|) \|r\|$$

Solution: $Ax = b$, $r = b - A\hat{x}$

$Ax - A\hat{x} = r \implies x - \hat{x} = A^{-1}r$

$\implies \|x - \hat{x}\| \leq \|A^{-1}\| \, \|r\|$.

$\|b\| \leq \|A\| \, \|x\| \implies \dfrac{1}{\|x\|} \leq \dfrac{\|A\|}{\|b\|}$.

Thus $\dfrac{\|x - \hat{x}\|}{\|x\|} \leq \left(\|A\| \, \|A^{-1}\| \right) \dfrac{\|r\|}{\|b\|}$

$$Ax = b, \quad r = b - A\hat{x}$$

$$\Rightarrow Ax - A\hat{x} = r \Rightarrow \|r\| \leq \|A\| \, \|x - \hat{x}\|$$

$$Ax = b \Rightarrow x = A^{-1}b \Rightarrow \|x\| \leq \|A^{-1}\| \, \|b\|$$

Hence

$$\left(\frac{1}{\|A\| \, \|A^{-1}\|}\right) \frac{\|r\|}{\|b\|} \leq \frac{\|x - \hat{x}\|}{\|x\|}$$

Q.4. Let $A(\alpha) = \begin{bmatrix} 0.1\alpha & 0.1\alpha \\ 1.0 & 2.5 \end{bmatrix}$.

Determine α so that the condition number of $A(\alpha)$ with respect to the ∞ norm is minimized.

Solution:

$$A(\alpha) = \begin{bmatrix} 0.1\alpha & 0.1\alpha \\ 1.0 & 2.5 \end{bmatrix}$$

$$\| A(\alpha) \|_\infty = \max \{ 0.2|\alpha|, 3.5 \}$$

$$A(\alpha)^{-1} = \frac{1}{0.15\alpha} \begin{bmatrix} 2.5 & -0.1\alpha \\ -1.0 & 0.1\alpha \end{bmatrix}$$

$$\| A(\alpha)^{-1} \|_\infty = \max \left\{ \frac{2.5 + 0.1|\alpha|}{0.15|\alpha|}, \frac{1.0 + 0.1|\alpha|}{0.15|\alpha|} \right\}$$

$$= \frac{2.5 + 0.1|\alpha|}{0.15|\alpha|} = \frac{50}{3|\alpha|} + \frac{2}{3}$$

$$\|A(\alpha)\|_\infty = \max\{0.2|\alpha|, 3.5\}$$

$$\|A(\alpha)^{-1}\|_\infty = \frac{50}{3|\alpha|} + \frac{2}{3}$$

Case 1) $0.2|\alpha| \leq 3.5 \Rightarrow |\alpha| \leq \frac{35}{2} = 17.5$

$\|A(\alpha)\|_\infty \|A(\alpha)^{-1}\|_\infty = 3.5\left(\frac{50}{3|\alpha|} + \frac{2}{3}\right)$: Minimum for $\underline{|\alpha| = 17.5}$

Case 2) $0.2|\alpha| \geq 3.5 \Rightarrow |\alpha| \geq 17.5$

$$\|A(\alpha)\|_\infty \|A(\alpha)^{-1}\|_\infty = 0.2|\alpha|\left(\frac{50}{3|\alpha|} + \frac{2}{3}\right)$$

$= \frac{10}{3} + \frac{0.4}{3}|\alpha|$: Minimum for $|\alpha| = 17.5$