Note Title 4/13/2011

Initial Value Problem

$$y' = f(x, y(x)), \quad x \in [a, b]$$

$$y(a) = y_0$$

y : unknown function to be determined

$$y' = \frac{dy}{dx}$$

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$$y'(x) = f(x, y(x)), \quad x \in [a, b]$$

$$y(a) = y_0 .$$

Let $x_0 = a$.

If y is twice-differentiable, then

$$y(x) = y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(c_x)$$

$$= y_0 + (x - x_0)\, f(x_0, y_0) + \frac{(x-x_0)^2}{2} y''(c_x)$$

$$a = x_0 < x_1 < \cdots < x_N = b$$

$$x_{n+1} - x_n = h = \frac{b-a}{N}, \quad n = 0, 1, \cdots, N-1$$

$y_n \simeq y(x_n) \longrightarrow$ exact value

$\downarrow$

approximation

$$y_0 = y(x_0)$$

Euler's Method

$$y' = f(x, y), \quad x \in [a, b],$$

$$y(x_0) = y(a) = y_0$$

$$a = x_0 < x_1 < \cdots < x_N = b, \quad h = \frac{b-a}{N}.$$

$$y(x) \simeq y(x_0) + (x - x_0)\, y'(x_0)$$

$$y(x_1) \simeq y_0 + h\, f(x_0, y_0)$$

$$y_{n+1} = y_n + h\, f(x_n, y_n), \quad n = 0, 1, 2, \cdots, N-1$$

$$y(x_{n+1}) = y(x_n) + h\, y'(x_n) + \frac{h^2}{2} y''(c_n)$$

$$= y(x_n) + h\, f(x_n, y(x_n)) + \frac{h^2}{2} y''(c_n)$$

<u>Euler's</u> <u>Method</u>

$$y_{n+1} = y_n + h\, f(x_n, y_n)$$

$$e_n = y(x_n) - y_n$$

$$e_{n+1} = e_n + h\left\{ f(x_n, y(x_n)) - f(x_n, y_n) \right\} + \frac{h^2}{2} y''(c_n)$$

$$e_{n+1} = e_n + h\left\{ f(x_n, y(x_n)) - f(x_n, y_n) \right\} + \frac{h^2}{2} y''(c_n), \quad c_n \in (x_n, x_{n+1})$$

$$f(x_n, y(x_n)) - f(x_n, y_n) = f_y(x_n, \bar{y}_n)(y(x_n) - y_n)$$

<u>MVT</u>

$$e_{n+1} = e_n + h f_y(x_n, \bar{y}_n) e_n + \frac{h^2}{2} y''(c_n),$$

$\bar{y}_n$ between $y(x_n)$ and y_n

$$e_{n+1} = e_n + h f_y(x_n, \bar{y}_n) + \frac{h^2}{2} y''(c_n)$$

Assume that.

$$|f_y(x, y)| \le L, \quad (x, y) \in D \quad \text{and}$$

$$|y''(x)| \le Y, \quad x \in [a, b].$$

Then

$$|e_{n+1}| \le (1 + hL)|e_n| + \frac{h^2}{2} Y.$$

$$e_n = y(x_n) - y_n$$

$$|e_{n+1}| \leq (1 + hL)|e_n| + \frac{h^2}{2} Y$$

<u>Claim:</u> $|e_n| \leq \frac{hY}{2L} \left\{ e^{(x_n - x_0)L} - 1 \right\}$

If x_n is kept fixed, then

$|e_n| \leq C h$: <u>discretization</u> <u>error</u>

$$|e_{n+1}| \le (1+hL)|e_n| + \frac{h^2}{2} Y \quad , \quad e_0 = 0$$

$$\xi_{n+1} = (1+hL)\xi_n + \frac{h^2}{2} Y \quad , \quad \xi_0 = 0$$

Claim: $|e_n| \le \xi_n$, $n = 0, 1, 2, \cdots$

Claim is true for $n = 0$.

Assume that $|e_k| \le \xi_k$. Then

$$|e_{k+1}| \le (1+hL)|e_k| + \frac{h^2}{2} Y$$

$$= (1+hL)\xi_k + \frac{h^2}{2} Y = \xi_{k+1}$$

$$\xi_{n+1} = (1+hL)\,\xi_n + \frac{h^2}{2} Y, \quad \xi_0 = 0$$

$$= (1+hL)^2 \xi_{n-1} + \frac{h^2}{2}\left[1 + (1+hL)\right] Y$$

$$= \cdots$$

$$= (1+hL)^{n+1} \xi_0 + \frac{h^2}{2}\left[1 + (1+hL) + \cdots + (1+hL)^n\right] Y$$

$$= \frac{h^2}{2}\left[\frac{(1+hL)^{n+1} - 1}{1+hL-1}\right] Y = \frac{hY}{2L}\left[(1+hL)^{n+1} - 1\right]$$

$$\xi_{n+1} = \frac{h Y}{2L}\left\{(1+hL)^{n+1} - 1\right\}$$

$$e^{x} = 1 + x + \frac{x^2}{2} e^{c}, \quad c \text{ between } 0 \text{ and } x.$$

$$\Rightarrow e^{x} \geq 1 + x \Rightarrow e^{nx} \geq (1+x)^{n}$$

$$|e_n| \leq \xi_n \leq \frac{hY}{2L}\left\{e^{nhL} - 1\right\}$$

$$= \frac{hY}{2L}\left\{e^{(x_n - x_0)L} - 1\right\}$$

Taylor's Method

$$y(x) = y(x_0) + (x - x_0)\, y'(x_0) + \cdots + \frac{(x-x_0)^k}{k!}\, y^{(k)}(x_0)$$

$$+ \frac{(x-x_0)^{k+1}}{(k+1)!}\, y^{(k+1)}(c_x), \quad c_x \text{ between } x \text{ and } x_0.$$

$$y'(x) = f(x, y(x))$$

$$y'' = f_x + f_y\, y' = f_x + f_y\, f$$

$$y' = f(x, y) \quad , \quad y'' = f_x + f_y f$$

$$y''' = f_{xx} + f_{xy} f + f_{yx} f + f_{yy} (f)^2 + f_y f_x + (f_y)^2 f$$

$$f_{xy} = f_{yx}$$

Taylor's Algorithm of order k

Define

$$T_k(x,y) = f(x,y) + \frac{h}{2} f'(x,y) + \cdots + \frac{h^{k-1}}{k!} f^{(k)}(x,y)$$

$$y_{n+1} = y_n + h\, T_k(x_n, y_n), \quad n = 0, 1, 2, \cdots, N-1$$

$k = 1$: Euler's Method

$$y(x_{n+1}) = y(x_n) + h\,y'(x_n) + \cdots + \frac{h^k}{k!}\,y^{(k)}(x_n) + \frac{h^{k+1}}{(k+1)!}\,y^{(k+1)}(c_n)$$

$$= y(x_n) + h\,T_k(x_n, y(x_n)) + \frac{h^{k+1}}{(k+1)!}\,y^{(k+1)}(c_n)$$

$$e_{n+1} = e_n + h\left[T_k(x_n, y(x_n)) - T_k(x_n, y_n)\right] + \frac{h^{k+1}}{(k+1)!}\,y^{(k+1)}(c_n)$$

<u>local discretization error</u>

Rungé - Kutta Method of order 2

$$y(x_{n+1}) = y(x_n) + h\, y'(x_n) + \frac{h^2}{2} y''(x_n) + \frac{h^3}{6} y'''(x_n) + O(h^4) \quad \cdots (1)$$

$$y_{n+1} = y_n + a k_1 + b k_2, \quad \cdots (2)$$

$$k_1 = h f(x_n, y_n), \quad k_2 = h f(x_n + \alpha h, y_n + \beta k_1)$$

Aim: To determine a, b, α, β such that (2) agrees with (1) for highest possible order