Note Title 4/13/2011

Initial Value Problem

$$y' = f(x, y(x)), \quad x \in [a, b]$$

$$y(a) = y_0$$

y : unknown function to be determined

$$y' = \frac{dy}{dx}$$

$$a = x_0 < x_1 < \cdots < x_N = b$$

$$x_{n+1} - x_n = h = \frac{b-a}{N},$$

$$n = 0, 1, \ldots, N-1$$

$y_n \simeq y(x_n) \longrightarrow$ exact value

$\downarrow$

approximation

Define

$$g(x) = f(x, y(x)), \quad x \in [a, b]$$

Then

$$g'(x) = f_x(x, y(x)) + f_y(x, y(x))\, y'(x)$$

$$= \left(f_x + f_y f\right)(x, y(x))$$

$$g'(x) = (f_x + f_y\, f)(x, y(x))$$

$$g''(x) = (f_{xx} + f_{xy}\, f + f_{yx}\, f$$
$$+ f_{yy}\,(f)^2 + f_y\, f_x + (f_y)^2 f)$$

Note Title 4/15/2011

Rungé - Kutta Method of order 2

$$y(x_{n+1}) = y(x_n) + h\, y'(x_n) + \frac{h^2}{2} y''(x_n) + \frac{h^3}{6} y'''(x_n) + O(h^4) \quad \cdots (1)$$

$$y_{n+1} = y_n + a k_1 + b k_2, \quad \cdots (2)$$

$$k_1 = h f(x_n, y_n),$$

$$k_2 = h f(x_n + \alpha h,\ y_n + \beta k_1)$$

$$y(x_{n+1}) = y(x_n) + h\, y'(x_n) + \frac{h^2}{2} y''(x_n)$$

$$+ \frac{h^3}{6} y'''(x_n) + O(h^4)$$

$$= y(x_n) + h\, f(x_n, y_n) + \frac{h^2}{2}(f_x + f_y \cdot f)$$

$$+ \frac{h^3}{6}\Big(f_{xx} + 2 f_{xy} f + f_{yy} (f)^2$$

$$+ f_y f_x + (f_y)^2 f\Big)(x_n, y(x_n))$$

$$y_{n+1} = y_n + a k_1 + b k_2$$

$$k_1 = h f(x_n, y_n)$$

$$k_2 = h f(x_n + \alpha h, y_n + \beta k_1)$$

$$f(x_n + \alpha h, \; y_n + \beta k_1)$$

$$= f(x_n, y_n) + (\alpha h \, f_x + \beta k_1 \, f_y)(x_n, y_n)$$

$$+ \left(\frac{(\alpha h)^2}{2} f_{xx} + (\alpha h)(\beta k_1) f_{xy} \right.$$

$$\left. + \frac{(\beta k_1)^2}{2} f_{yy} \right)(x_n, y_n)$$

$$+ O(h^3)$$

$$y_{n+1} = y_n + h(a+b) f(x_n, y_n)$$
$$+ h^2 b(\alpha f_x + \beta f_y f)(x_n, y_n)$$
$$+ b h^3 \left(\frac{\alpha^2}{2} f_{xx} + \alpha\beta f_{xy} f + \frac{\beta^2}{2} f_{yy} (f)^2\right)$$
$$+ O(h^4)$$

$$y(x_{n+1}) = y(x_n) + h f(x_n, y(x_n))$$
$$+ \frac{h^2}{2}(f_x + f_y \cdot f)(x_n, y(x_n))$$
$$+ \frac{h^3}{6}(f_{xx} + 2 f_{xy} f + f_{yy}(f)^2 + f_x f_y + (f_y)^2 f)$$
$$+ O(h^4)$$

$$y_{n+1} = y_n + a k_1 + b k_2,$$

$$k_1 = h f(x_n, y_n)$$

$$k_2 = h f(x_n + \alpha h, y_n + \beta k_1)$$

$$a + b = 1, \quad b\alpha = \frac{1}{2}, \quad b\beta = \frac{1}{2}$$

$$a = b = \frac{1}{2}, \quad \alpha = \beta = 1$$

local discretization error $= O(h^3)$

$$y(x_{n+1}) - y_{n+1}$$

$$= \frac{h^3}{6}\left(f_{xx} + 2 f_{xy} f + f_{yy} (f)^2 + f_x f_y + (f_y)^2 f\right)$$

$$- \frac{h^3}{4}\left(f_{xx} + 2 f_{xy} f + f_{yy} (f)^2\right) + O(h^4)$$

$$= -\frac{h^3}{12}\left(f_{xx} + 2 f_{xy} f + f_{yy} (f)^2 - 2 f_x f_y - 2 (f_y)^2 f\right)$$

$$+ O(h^4)$$

Runge - Kutta Method of order 4

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

$$k_1 = h\, f(x_n, y_n),$$

$$k_2 = h\, f\left(x_n + \frac{h}{2},\ y_n + \frac{k_1}{2}\right),$$

$$k_3 = h\, f\left(x_n + \frac{h}{2},\ y_n + \frac{k_2}{2}\right),$$

$$k_4 = h\, f(x_n + h,\ y_n + k_3).$$

local error $= O(h^5)$

Numerical Integration

$$y'(x) = f(x, y(x)) = g(x), \quad x \in [a, b]$$

$$a = x_0 < x_1 < \cdots < x_N = b, \quad h = \frac{b-a}{N}$$

$$\int_{x_n}^{x_{n+1}} y'(x)\,dx = \int_{x_n}^{x_{n+1}} f(x, y(x))\,dx$$

$$\parallel$$

$$y(x_{n+1}) - y(x_n)$$

Rectangle Rule.

$$g(x) = g(x_n) + g[x_n, x](x - x_n),$$
$$x \in [x_n, x_{n+1}]$$

$$\int_{x_n}^{x_{n+1}} g(x)\,dx \simeq h\, g(x_n)$$

$$\text{Error} = \int_{x_n}^{x_{n+1}} g[x_n, x](x - x_n)\,dx$$
$$= \frac{g'(c_n)\, h^2}{2}$$

$$y(x_{n+1}) - y(x_n) = \int_{x_n}^{x_{n+1}} f(x, y(x))\,dx$$

$$y_{n+1} - y_n = h\, f(x_n, y_n)$$

$$y_{n+1} = y_n + h\, f(x_n, y_n)$$: Euler's Method

Error $= \dfrac{h^2}{2} g'(c_n) = \dfrac{h^2}{2} y''(c_n)$

$$y(x_{n+1}) - y(x_n) = \int_{x_n}^{x_{n+1}} g(x)\,dx$$

p_3 : polynomial of degree ≤ 3

interpolating g at

$x_n, x_{n-1}, x_{n-2}, x_{n-3}$

$$g(x) = g(x_n) + g[x_n, x_{n-1}](x - x_n)$$

$$+ g[x_n, x_{n-1}, x_{n-2}](x - x_n)(x - x_{n-1})$$

$$+ g[x_n, x_{n-1}, x_{n-2}, x_{n-3}](x - x_n)(x - x_{n-1})(x - x_{n-2})$$

$$+ g[x_n, x_{n-1}, x_{n-2}, x_{n-3}, x]\, \omega(x),$$

$$\omega(x) = (x - x_n)(x - x_{n-1})(x - x_{n-2})(x - x_{n-3})$$

$$g[x_n, x_{n-1}] = \frac{g(x_n) - g(x_{n-1})}{x_n - x_{n-1}}$$

$$= \frac{\Delta g_{n-1}}{h}$$

$$g[x_n, x_{n-1}, x_{n-2}] = \frac{g(x_n) - 2g(x_{n-1}) + g(x_{n-2})}{2h^2}$$

$$= \frac{\Delta^2 g_{n-2}}{2h^2}$$

$$g[x_n, x_{n-1}, x_{n-2}, x_{n-3}]$$

$$= \frac{g(x_n) - 3\,g(x_{n-1}) + 3\,g(x_{n-2}) - g(x_{n-3})}{6h^3}$$

$$= \frac{\Delta^3 g_{n-3}}{6h^3}$$

$$p_3(x) = g_n + \Delta g_{n-1} \frac{x - x_n}{h}$$

$$+ \Delta^2 g_{n-2} \frac{(x - x_n)(x - x_{n-1})}{2h^2}$$

$$+ \Delta^3 g_{n-3} \frac{(x - x_n)(x - x_{n-1})(x - x_{n-2})}{6h^3}$$

$$y(x_{n+1}) - y(x_n) = \int_{x_n}^{x_{n+1}} f(x, y(x))\, dx$$

$$y_{n+1} - y_n = \int_{x_n}^{x_{n+1}} p_3(x)\, dx$$

Substitute $s = \frac{x - x_n}{h}$

$$y_{n+1} - y_n = h \int_0^1 p_3(s)\, ds$$

$$p_3(x) = g_n + \Delta g_{n-1} \frac{x - x_n}{h} + \Delta^2 g_{n-2} \frac{(x - x_n)(x - x_{n-1})}{2h^2} + \Delta^3 g_{n-3} \frac{(x - x_n)(x - x_{n-1})(x - x_{n-2})}{6h^3}$$

$$p_3(s) = g_n + \Delta g_{n-1} s + \Delta^2 g_{n-1} \frac{s(s+1)}{2} + \Delta^3 g_{n-3} \frac{s(s+1)(s+2)}{6}$$

$$y_{n+1} - y_n = h \int_0^1 \Big\{ g_n + \Delta g_{n-1} s + \frac{\Delta^2 g_{n-2}}{2} s(s+1) + \frac{\Delta^3 g_{n-3}}{6} s(s+1)(s+2) \Big\} ds$$

$$= h \Big\{ g_n + \frac{\Delta g_{n-1}}{2} + \frac{\Delta^2 g_{n-2}}{2}\left(\frac{1}{3} + \frac{1}{2}\right) + \frac{\Delta^3 g_{n-3}}{6}\left(\frac{1}{4} + 1 + 1\right) \Big\}$$

$$= h \Big\{ g_n + \frac{\Delta g_{n-1}}{2} + \Delta^2 g_{n-2} \frac{5}{12} + \Delta^3 g_{n-3} \frac{9}{24} \Big\}$$

$$y_{n+1} - y_n =$$

$$h\left\{ g_n + \frac{\Delta g_{n-1}}{2} + \Delta^2 g_{n-2}\,\frac{5}{12} + \Delta^3 g_{n-3}\,\frac{9}{24} \right\}$$

$$= h\left\{ g_n + \frac{g_n - g_{n-1}}{2} + \frac{5\left(g_n - 2\,g_{n-1} + g_{n-2}\right)}{12} \right.$$

$$\left. + \frac{9\left(g_n - 3\,g_{n-1} + 3\,g_{n-2} - g_{n-3}\right)}{24} \right\}$$

$$= \frac{h}{24}\left\{ 55\,g_n - 59\,g_{n-1} + 37\,g_{n-2} - 9\,g_{n-3} \right\}$$

Adams-Bashforth Method

$$y_{n+1} = y_n + \frac{h}{24}\left[55\,g_n - 59\,g_{n-1} + 37\,g_{n-2} - 9\,g_{n-3}\right]$$

$$g_n = g(x_n) = f(x_n, y_n)$$

y_0, y_1, y_2, y_3 : given