Note Title 4/19/2011

Adams-Bashforth Method

$y'(x) = f(x, y(x)) = g(x), \; x \in [a, b]$

$y(a) = y_0$

$a = x_0 < x_1 < \cdots < x_N = b$

$h = \dfrac{b-a}{N}$

P_3 : polynomial of degree ≤ 3

interpolating g at $x_n, x_{n-1}, x_{n-2}, x_{n-3}$

$$y'(x) = f(x, y(x)) = g(x)$$

$$y(x_{n+1}) - y(x_n) = \int_{x_n}^{x_{n+1}} g(x)\,dx$$

Adams - Bashforth Method

$$y_{n+1} - y_n = \int_{x_n}^{x_{n+1}} g(x)\,dx$$

Note Title 4/18/2011

$$y_{n+1} = y_n + \frac{h}{24}\left[55\, g_n - 59\, g_{n-1} + 37\, g_{n-2} - 9\, g_{n-3}\right]$$

$$g_n = g(x_n) = f(x_n, y_n)$$

y_0, y_1, y_2, y_3 : given

Error

$$E_{AB} = \int_{x_n}^{x_{n+1}} g[x_n, x_{n-1}, x_{n-2}, x_{n-3}, x] (x-x_n)(x-x_{n-1})(x-x_{n-2})(x-x_{n-3}) \, dx$$

$$= \frac{251}{720} h^5 g^{(4)}(d_n)$$

$$= \frac{251}{720} y^{(5)}(d_n) h^5$$

In order to calculate y_1, y_2, y_3, use Rungé - Kutta method of order 4 which has local discretization error of $O(h^5)$.

Predictor- Corrector formulae

$$y(x_{n+1}) - y(x_n) = \int_{x_n}^{x_{n+1}} f(x, y(x))\,dx$$

Trapezoidal Rule

$$y_{n+1} - y_n = \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right]$$

$$\text{error} = -\frac{h^3}{12}\, y'''(\alpha_n)$$

Euler's Method

$$y_{n+1} = y_n + h\, f(x_n, y_n)$$

$$\text{error} = \frac{h^2}{2}\, y''(C_n)$$

Trapezoidal Rule

$$y_{n+1} = y_n + h\, \frac{\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right]}{2}$$

$$\text{error} = -\frac{h^3}{12}\, y'''(d_n)$$

Fix x_n. Define

$$y_{n+1}^{(0)} = y_n + h\, f(x_n, y_n)$$

$$y_{n+1}^{(1)} = y_n + h\, \frac{\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1}^{(0)}) \right]}{2},$$

$$y_{n+1}^{(k)} = y_n + h\, \frac{\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1}^{(k-1)}) \right]}{2}$$

ϵ is prescribed

Stop the iteration when

$$\frac{\left| y_{n+1}^{(k)} - y_{n+1}^{(k-1)} \right|}{\left| y_{n+1}^{(k)} \right|} < \epsilon .$$

Euler's formula : Open or Predictor formula

Trapezoidal Rule : Closed or Corrector formula

Corrector formula : More accurate than Predictor formula (Generally)

For the algorithm, one needs to specify

1) h 2) maximum number, K, of iterations

3) what to do if k exceeds K.

In practice, if h is properly chosen and if predictor and corrector formula are of the same order, then one or two iterations suffice.

Otherwise reduce h.

Adams - Moulton method.

$$y(x_{n+1}) - y(x_n) = \int_{x_n}^{x_{n+1}} f(x, y(x))\,dx$$

p_3 : polynomial of degree ≤ 3

interpolating $g(x) = f(x, y(x))$

at $x_{n+1}, x_n, x_{n-1}, x_{n-2}$

$$y_{n+1} - y_n = \int_{x_n}^{x_{n+1}} p_3(x)\,dx$$

$$= \frac{h}{24}\left(9\, g_{n+1} + 19\, g_n - 5\, g_{n-1} + g_{n-2}\right),$$

$$g_n = g(x_n) = f(x_n, y_n)$$

$$\text{Error} = -\frac{19}{720} h^5 y^{(5)}(d)$$

Adams-Moulton Predictor-Corrector Method

$y' = f(x, y)$, $y(x_0) = y_0$, $x_n = x_0 + nh$

$g_n = g(x_n) = f(x_n, y_n)$

y_0, y_1, y_2, y_3 : given

For $n = 3, 4, \cdots,$

$$y_{n+1}^{(0)} = y_n + \frac{h}{24}(55 g_n - 59 g_{n-1} + 37 g_{n-2} - 9 g_{n-3})$$

Adams-Bashforth (AB)

Adams - Moulton (AM)

$$y_{n+1}^{(k)} = y_n + \frac{h}{24}\left[9 f\left(x_{n+1}, y_{n+1}^{(k-1)}\right) + 19 g_n - 5 g_{n-1} + g_{n-2}\right]$$

Iterate on k until

$$\frac{\left| y_{n+1}^{(k)} - y_{n+1}^{(k-1)} \right|}{\left| y_{n+1}^{(k)} \right|} < e$$

$$E_{AB} = \frac{251}{720} y^{(5)}(c_n) h^5$$

$$E_{AM} = -\frac{19}{720} y^{(5)}(d_n) h^5$$

$$y(x_{n+1}) - y_{n+1}^{(0)} = \frac{251}{720} y^{(5)}(c_n) h^5$$

$$y(x_{n+1}) - y_{n+1}^{(1)} = \frac{-19}{720} y^{(5)}(d_n) h^5$$

$$y_{n+1}^{(1)} - y_{n+1}^{(0)} \simeq \frac{270}{720} y^{(5)}(d_n) h^5$$

$$y(x_{n+1}) - y_{n+1}^{(1)} = \frac{-19}{720} y^{(5)}(d_n) h^5$$

$$y_{n+1}^{(1)} - y_{n+1}^{(0)} \simeq \frac{270}{720} y^{(5)}(d_n) h^5$$

$$y(x_{n+1}) - y_{n+1}^{(1)} \simeq -\frac{19}{270}\left(y_{n+1}^{(1)} - y_{n+1}^{(0)}\right)$$

$$\simeq -\frac{1}{14}\left(y_{n+1}^{(1)} - y_{n+1}^{(0)}\right)$$

$$= D_{n+1}$$

$\dfrac{|D_{n+1}|}{h}$: local error per unit step

We want $E_1 < \dfrac{|D_{n+1}|}{h} < E_2$

If $E_1 < \dfrac{|D_{n+1}|}{h} < E_2$, proceed to y_{n+2} with the same step h.

If $\dfrac{|D_{n+1}|}{h} > E_2$, reduce h to $\dfrac{h}{2}$

If $\dfrac{|D_{n+1}|}{h} < E$, increase h to $2h$

Comparison

RK method of order 4	AB	AM
$O(h^5)$	$O(h^5)$	$O(h^5)$
self-starting	x	x
4 function evaluation /step	1	2
		error control

$$y'(x) = f(x, y(x)) = g(x), \quad x \in [a, b]$$

$$y(a) = y_0$$

$$\int_{x_{n-1}}^{x_{n+1}} y'(x)\,dx = \int_{x_{n-1}}^{x_{n+1}} g(x)\,dx$$

<u>Midpoint Rule</u>

$$y_{n+1} - y_{n-1} = 2h\, g(x_n) = 2h\, f(x_n, y_n)$$

Midpoint Rule

$$y_{n+1} = y_{n-1} + 2h\, f(x_n, y_n)$$

$$\text{Error} = \frac{h^3}{3} y'''(c_n)$$

Euler's Method

$$y_{n+1} = y_n + h\, f(x_n, y_n)$$

$$\text{Error} = \frac{h^2}{2} y''(d_n)$$

$y' = -2y + 1$, $y(0) = 1$ (Conté - de - Boor)

exact solution : $y(x) = \frac{1}{2} e^{-2x} + \frac{1}{2}$.

$[0, 4]$, $h = \frac{1}{32}$.

x_n	Euler	Midpoint
0	0	0
0.5	−0.00590	0.000142
1.0	−0.00427	0.000157
1.5	−0.00232	0.000239
3.0	−0.00022	0.003836
4.0	−0.000038	0.02827

Stability

$$y' = \lambda y, \quad y(0) = 1$$

Exact Solution : $y(x) = e^{\lambda x}$, $x \in [0, b]$

$$h = \frac{b}{N}, \quad x_n = nh$$

Euler's Method :
$$\begin{aligned} y_{n+1} &= y_n + h\lambda y_n \\ &= (1 + h\lambda) y_n \\ &= (1 + h\lambda)^{n+1} \end{aligned}$$

$$e^{x} = 1 + x + \frac{x^2}{2} e^{c}$$

$$= 1 + x + O(x^2)$$

$$e^{nx} = (1+x)^{n} + O(x^2)$$

$$e^{nh\lambda} = (1+h\lambda)^{n} + O(h^2\lambda^2)$$

$$y(x_n) - y_n = O(h^2\lambda^2).$$